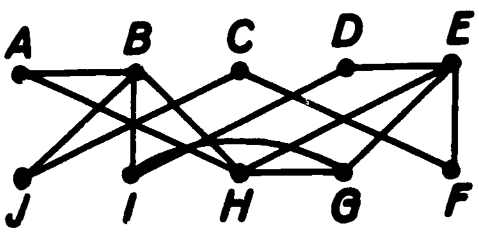
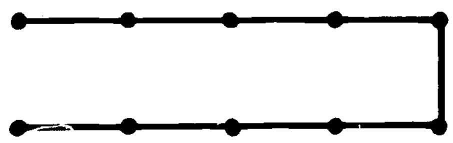
Bonus Puzzle 1: A city planner, after pondering over the Koenigsberg Bridge puzzle, commented "I could have relocated any one of the existing bridges and made it easy for the burghers to take their promenade." Explain why the city planner could comment thusly.

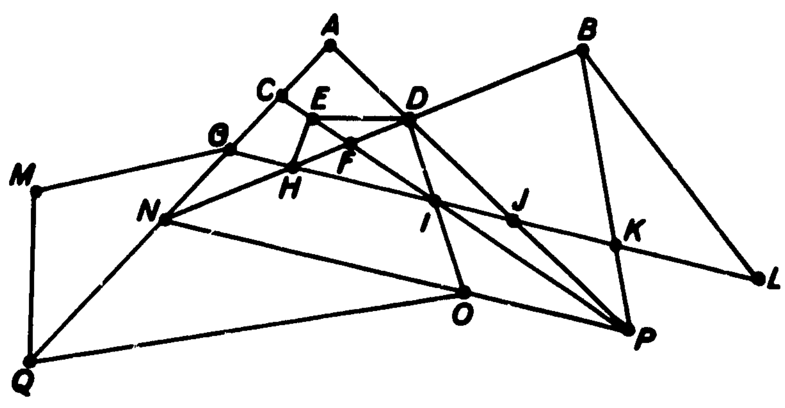
Bonus Puzzle 2: A basketball league consists of 7 teams who play a round robin against each other. If each team plays each of the other 6 teams once, is it possible to have an exact 7-way tie between the teams at the end of the 21 game league season? If so, describe a possible sequence of wins and losses between the teams that will result in this 7-way tie.

Bonus Puzzle 3: A 10-pin connecting terminal is wired as shown at the left for a special piece of electrical equipment. An industrial engineer claims that the simple wiring shown at the right will do, using the existing connections. He claims that all we need do is to rename the terminals and, in fact, we can do away with five of the 14 wire connections. How did he rename the pins?

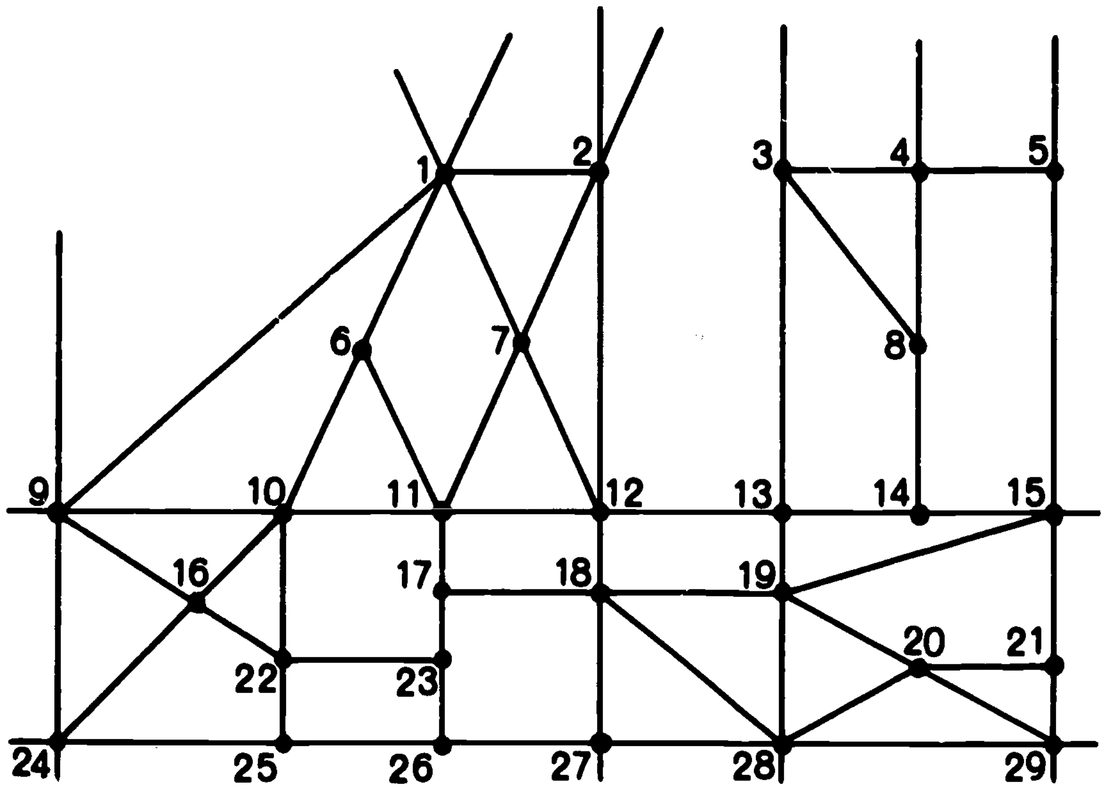




Bonus Puzzle 4: A sanitation engineer has the unpleasant task of inspecting every foot of an underground drainage system by crawling through it. Since climbing into and out of the system is difficult, he wants to make as few entries and exits as possible while inspecting each section exactly once. Refer to the drawing below to determine how many trips he must make and where he should enter and exit from the system.

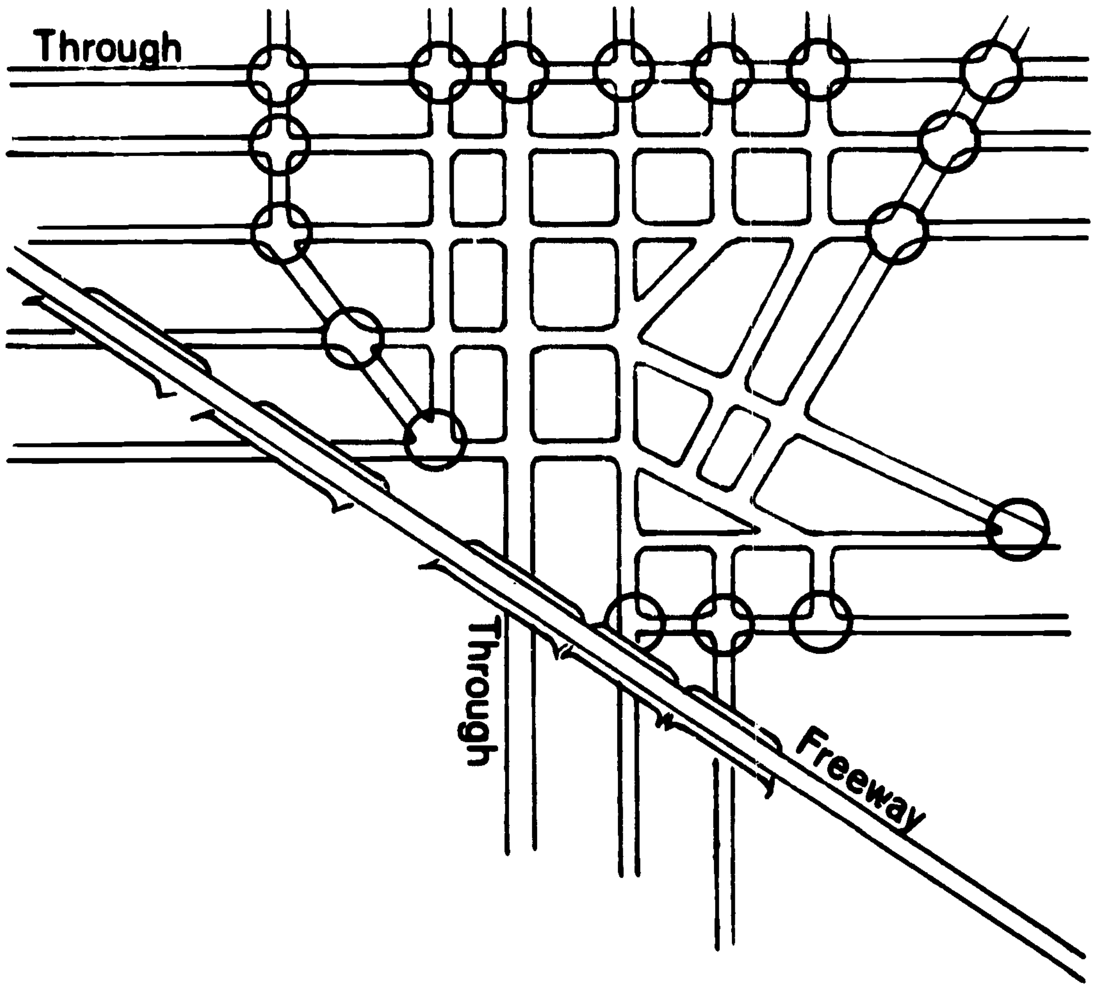


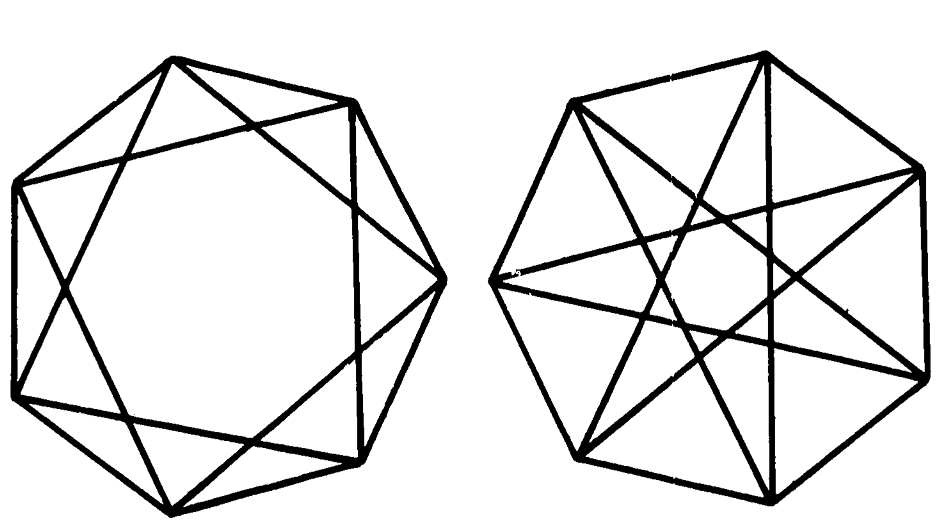
Bonus Puzzle 5: A young man began a paper route with 29 "corner" customers. Before his first delivery, he was given a map with the location of customers shown at the numbered corners. Wishing to be efficient, he decided to plan his route so that he could start at any customer's home and deliver papers to each succeeding customer without passing any customer's home twice and end his deliveries at the initial customer's home. Can you plan the route for the young man?



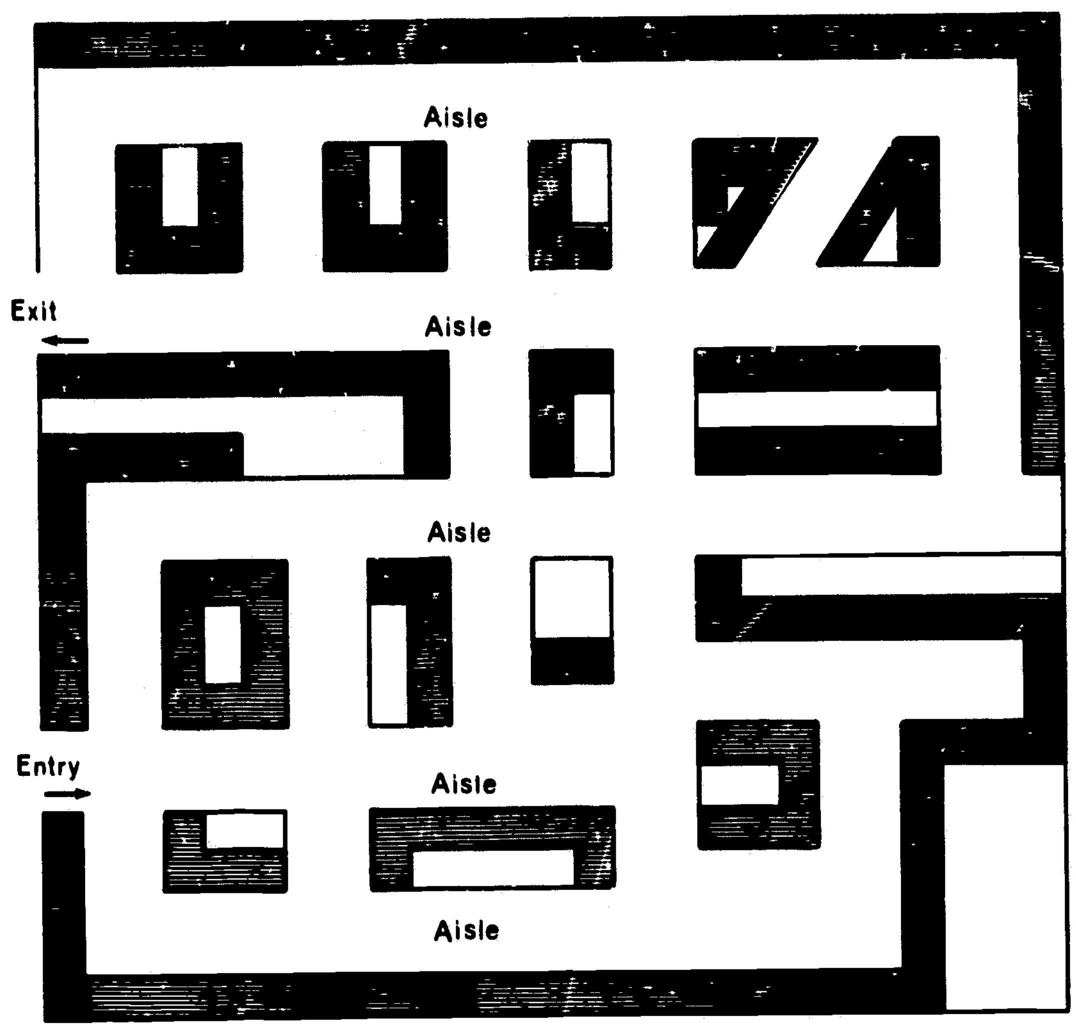
Bonus Puzzle 6: A city council decided to convert all possible streets in a business section into strictly one-way streets except for the two existing boulevards and the freeway over which they had no control. The business section is shown in the map on the opposite page. Direct all the possible streets bounded by the circled intersections so that a motorist can travel the streets in a reasonable manner and go from any intersection within the area to any other intersection within the area without going wrong-way on a street.

Bonus Puzzle 7: Show that the heptagons, opposite, are isomorphic.


Through
Through
Freeway



Bonus Puzzle 8: Sally and Bill were given a chance to visit a distant museum. Not wishing to miss any exhibits or waste time in the museum, they obtained a map of the museum. Question: Can they plan a single tour of the museum such that they would see all of the exhibits in one continuous trip without missing or revisiting any of the exhibits from entry to exit? If so, plan their tour. Note: The shaded parts of the map are the exhibits facing the aisles.



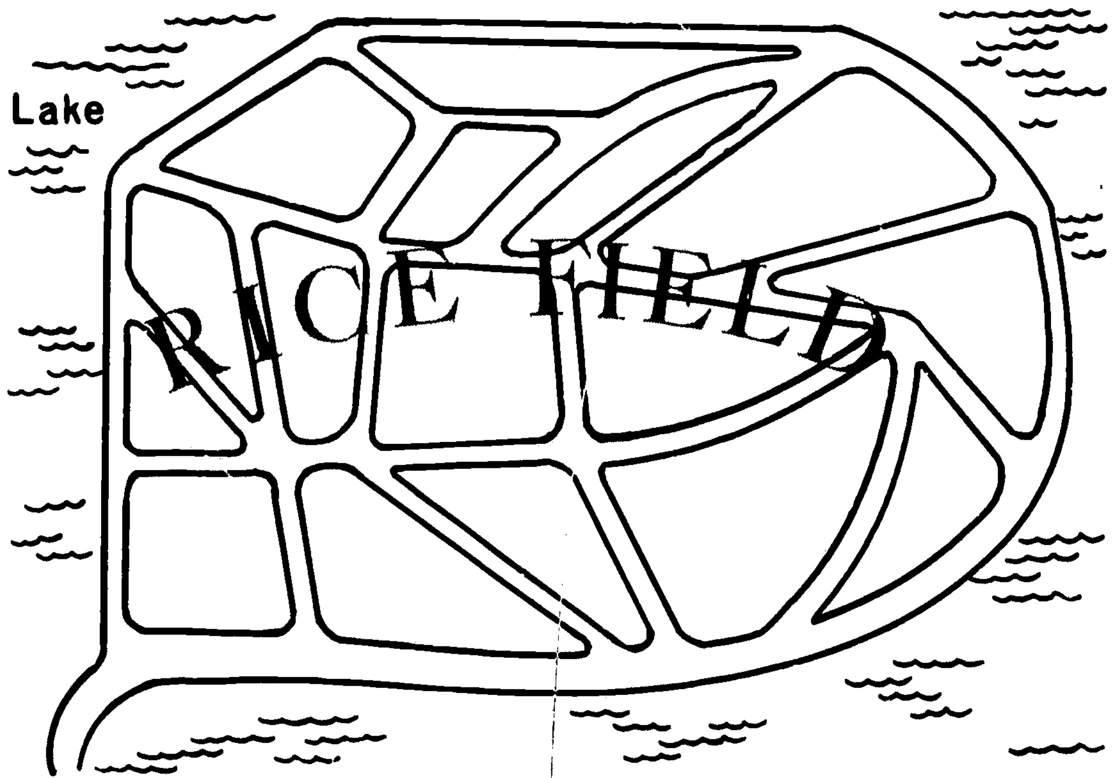
Bonus Puzzle 9: Mr. Jones wants to install a set of 6 connected sprinklers in his garden, using one water outlet. Planning to do all of the work himself, he determines that each sprinkler head will cost him \$2.25 and the pipe connections he may possibly require will cost him, in materials, the amounts listed in the table below. What arrangement of connections will result in a minimal cost for the material and what will this minimal cost be?

TABLE 2

Connection	Cost	Connection	Cost	Connection	Cost
WO* to SH† 1	$7.00	SH 1 to SH 3	$4.25	SH 2 to SH 6	$7.75
WO to SH 2	8.75	SH 1 to SH 4	4.00	SH 3 to SH 4	6.25
WO to SH 3	7.50	SH 1 to SH 5	3.50	SH 3 to SH 5	6.50
WO to SH 4	6.25	SH 1 to SH 6	6.50	SH 3 to SH 6	6.50
WO to SH 5	9.00	SH 2 to SH 3	4.00	SH 4 to SH 5	4.25
WO to SH 6	8.00	SH 2 to SH 4	4.25	SH 4 to SH 6	5.50
SH 1 to SH 2	$3.75	SH 2 to SH 5	$5.75	SH 5 to SH 6	$5.25

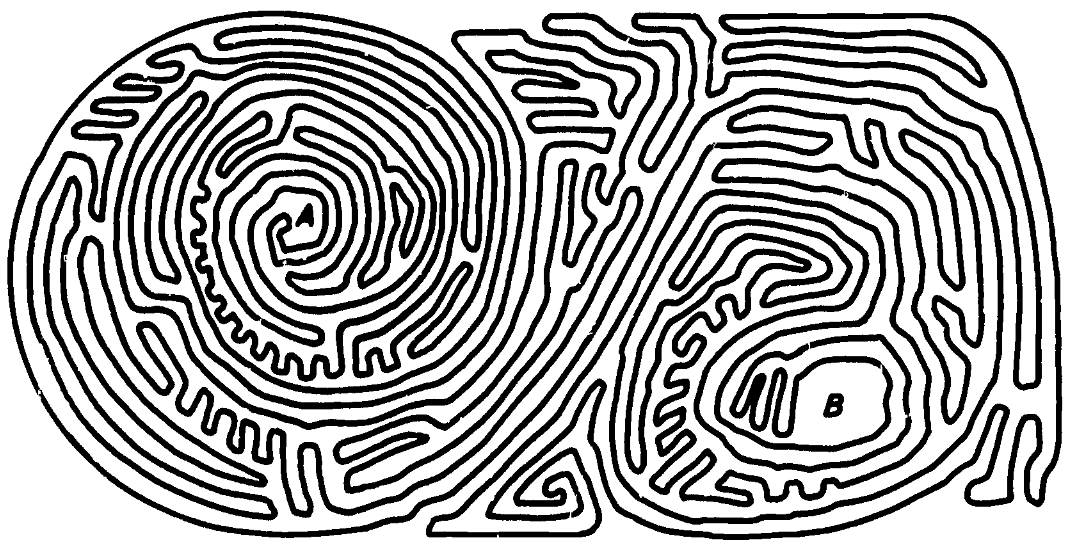
* Water Outlet
† Sprinkler Head

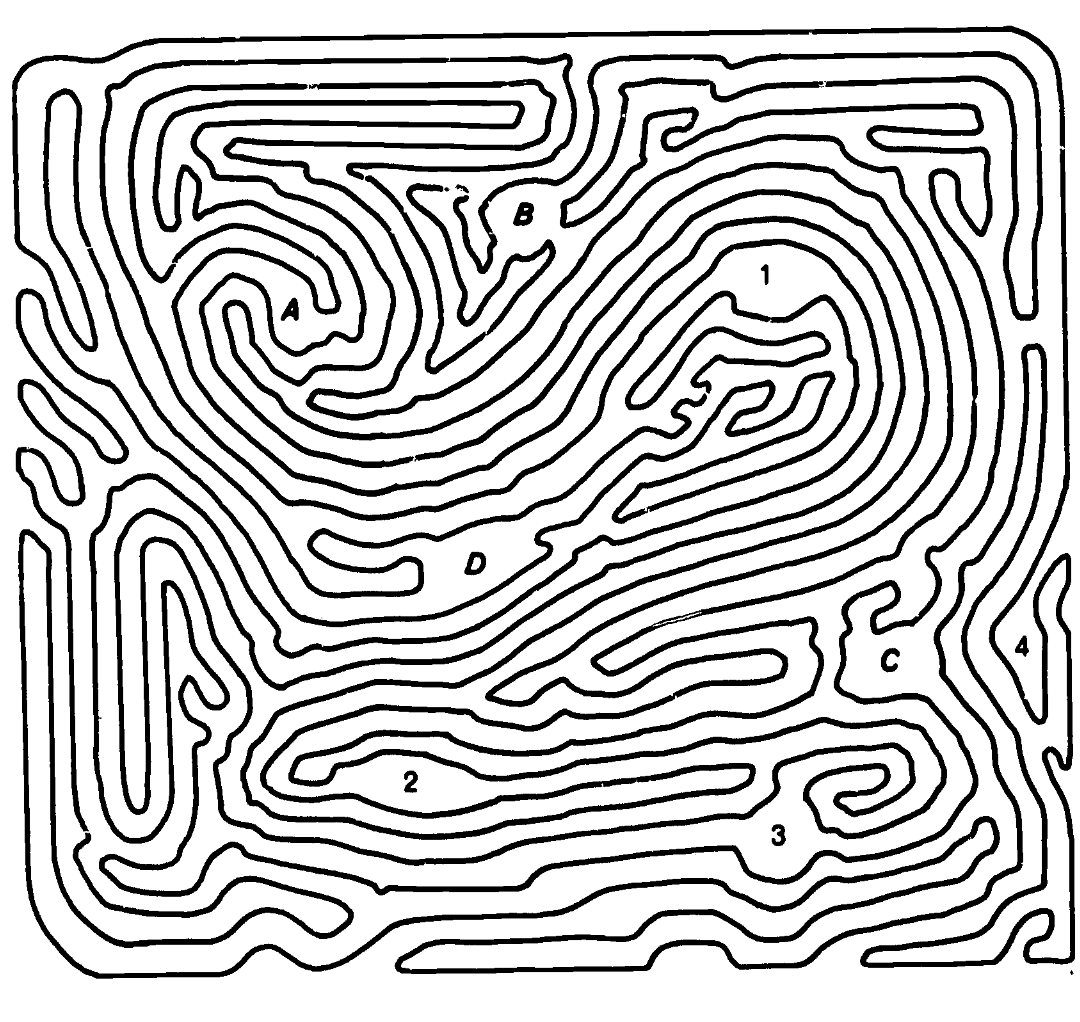
Bonus Puzzle 10: A rice farmer has his rice fields laid out as shown below. As is usual in rice cultivation, the rice fields are set low and surrounded by earthen dikes. The entire farm is surrounded by a lake



and in cultivation a dike is broken to immerse a field in water from the lake. The farmer would like to know what is the least number of dikes he must break in order to immerse all of his fields of rice.

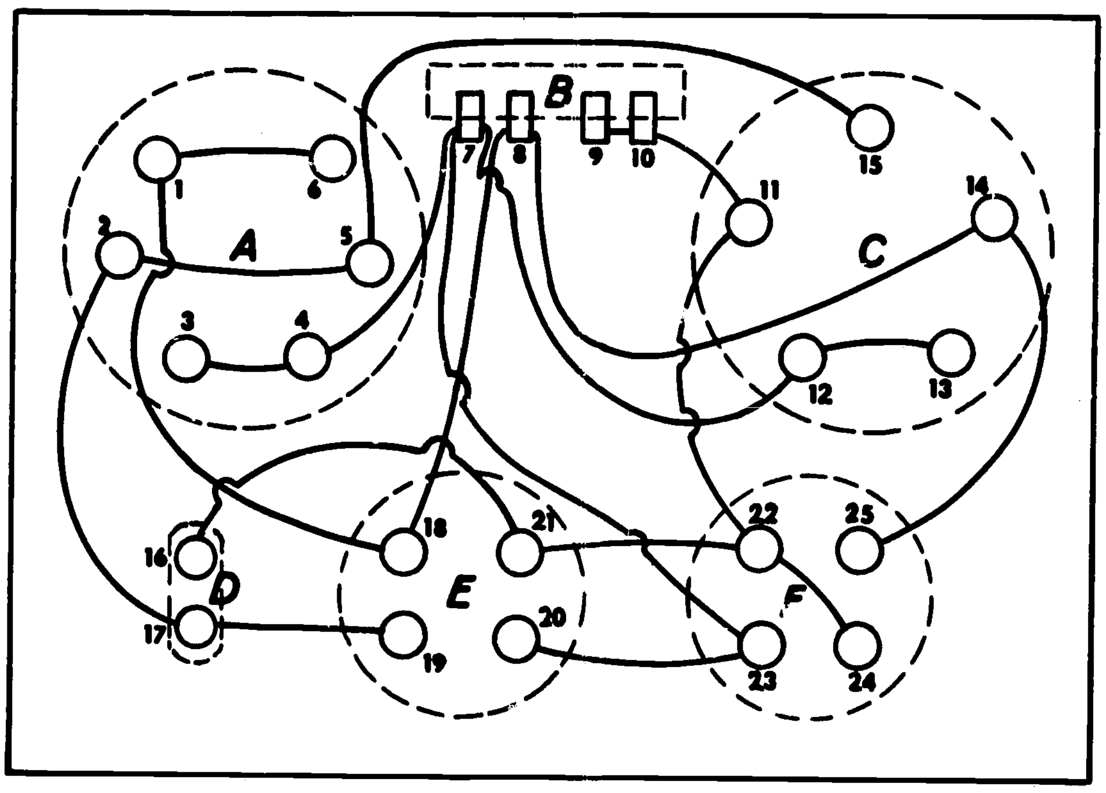
Bonus Puzzle 11: Can you draw a single path from point *A* to point *B* without crossing the curve or lifting the "marker"?





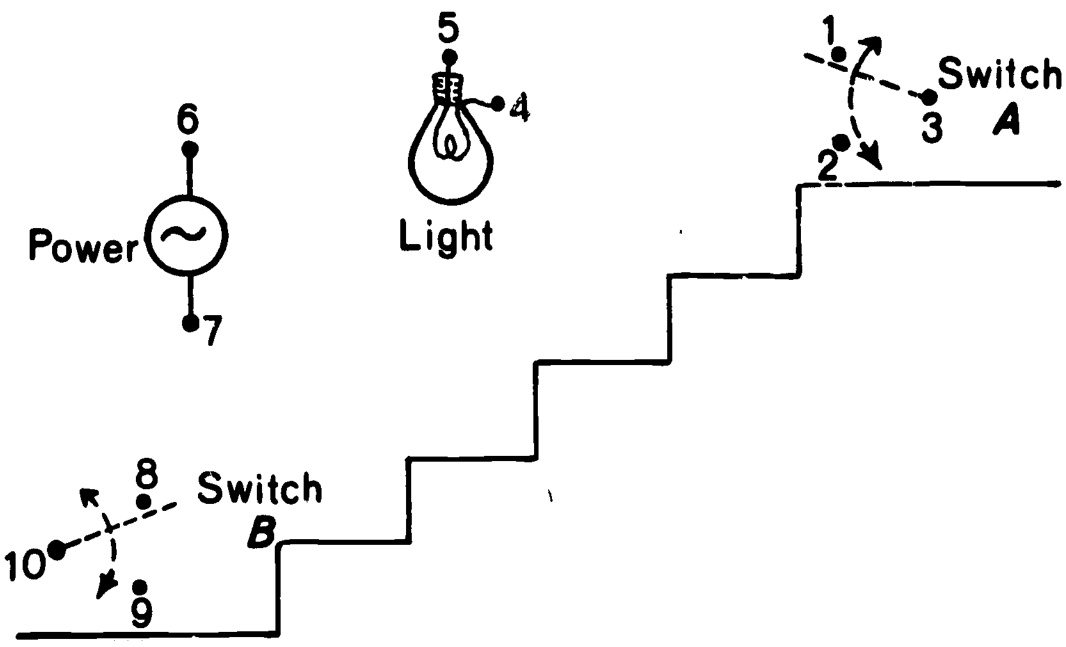
Bonus Puzzle 12: Three continuous closed curves wind about, as shown. Match the four lettered points to the four numbered points by drawing paths connecting a lettered point to a numbered point without crossing the curves: that is, find the letter-number pairs in each of the four regions formed by the three curves.

Bonus Puzzle 13: The base of an electrical instrument is to be redesigned as a printed circuit. The design engineer is given the diagram of the old base, wired as shown. He must design the new circuit so that the etching and plating for the printed circuit can be done without causing short circuits while retaining the arcs shown in the given base. If the plugs and two terminals, shown in dashed lines, can be relocated, can he design the printed circuit as required or must the design of the base wiring be changed?

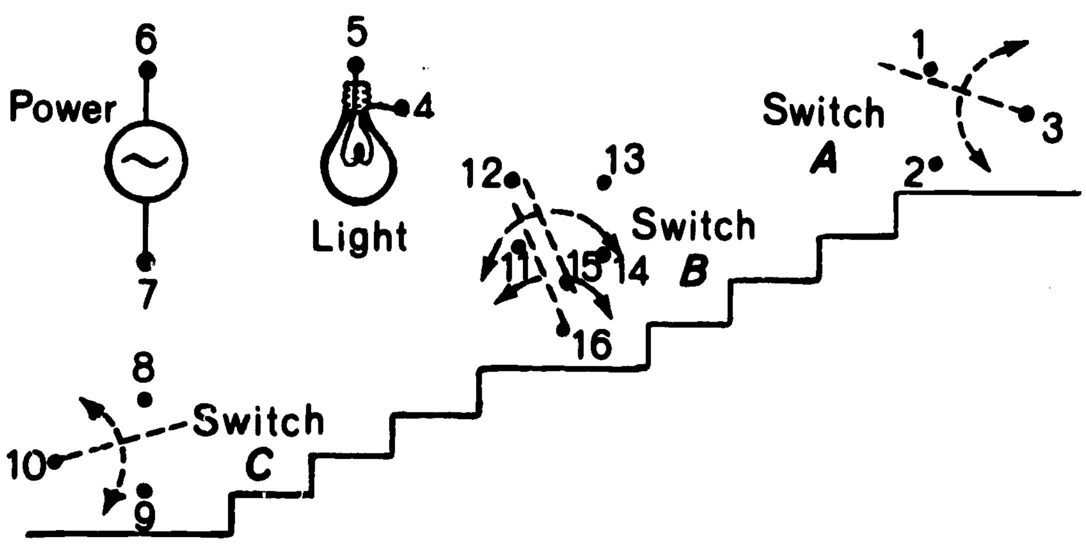


Bonus Puzzle 14: A man wishes to install a stairwell light and connect it with two switches so that he will be able to turn the light on or

off from either switch. Each switch has two positions with two circuits possible; when one circuit is closed the other is open, and vice versa. There are ten points of connection. How should he connect the ten points between the light, power, and switches?

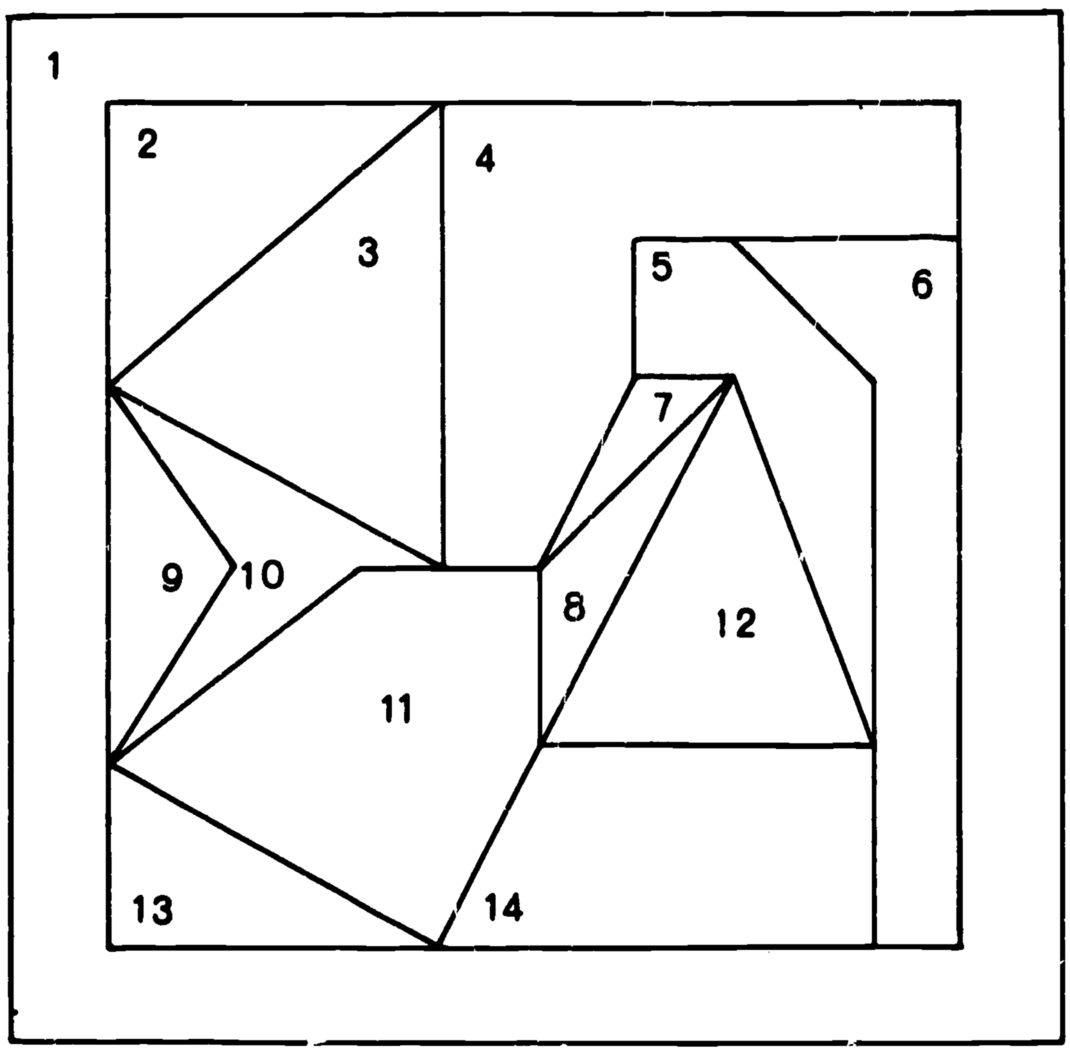


Bonus Puzzle 15: A man wishes to install a stairwell light and connect it with three switches so that he will be able to turn the light on or off from any one of the switches. Two of the switches are single pole switches having two positions with two circuits possible; when one circuit is open the other is closed, and vice versa. The third switch is a double pole switch having two positions with four possible circuits; the

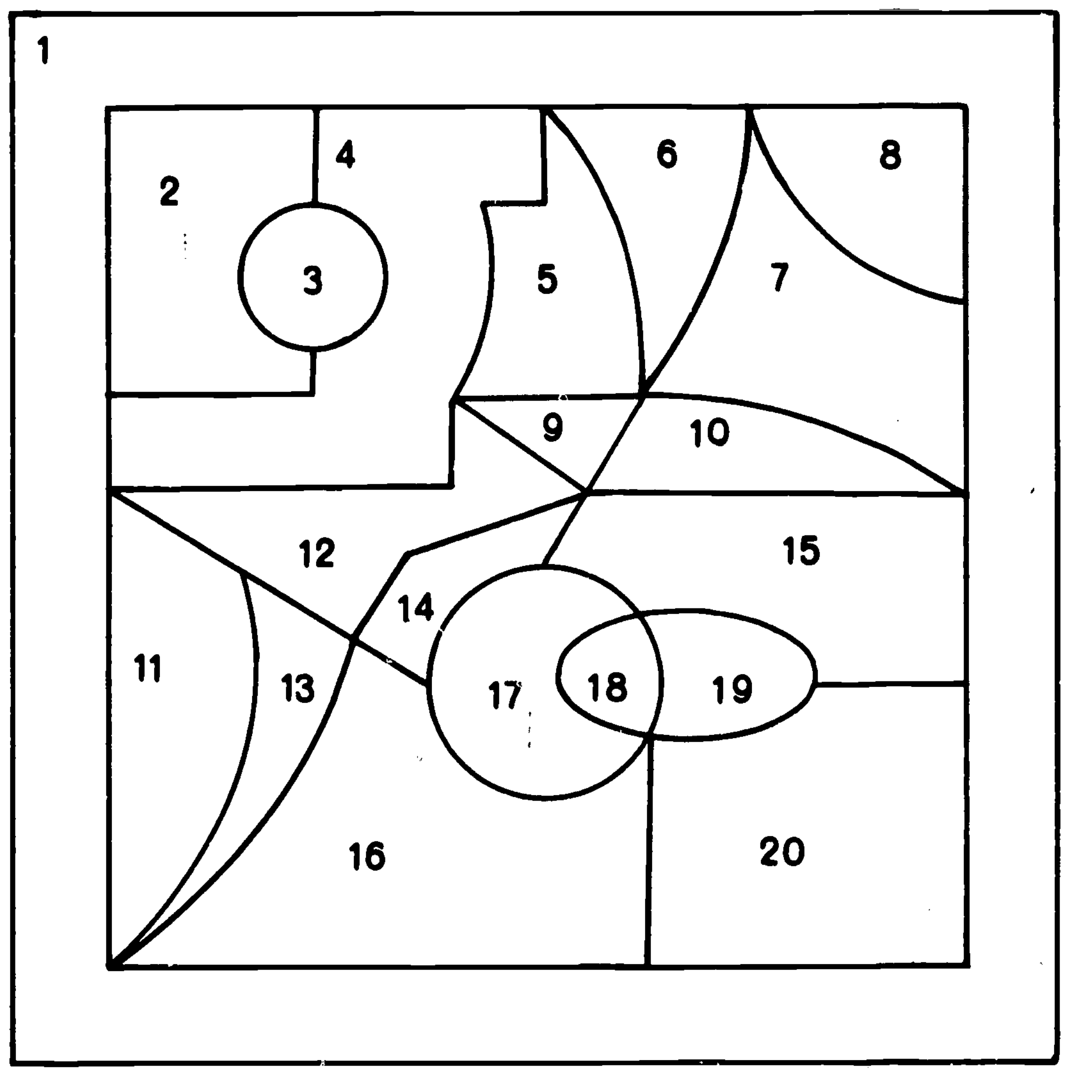


circuits are paired so that when two circuits are open the other two are closed, and vice versa. There are 16 points for connections. How should he connect the 16 points between the light, power, and switches?

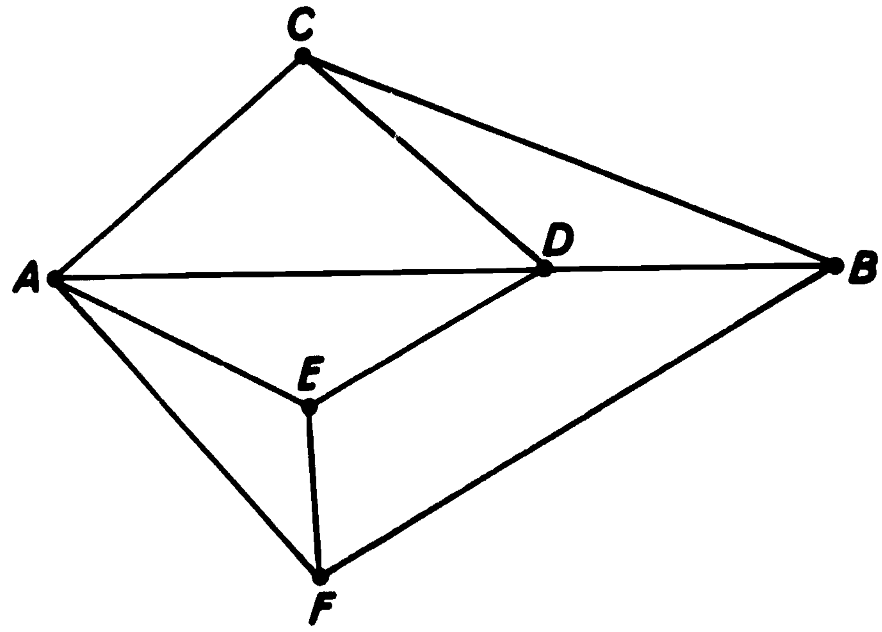
Bonus Puzzle 16: Color the "picture," including the border, using just three distinct colors so that no two faces with a common edge have the same color.



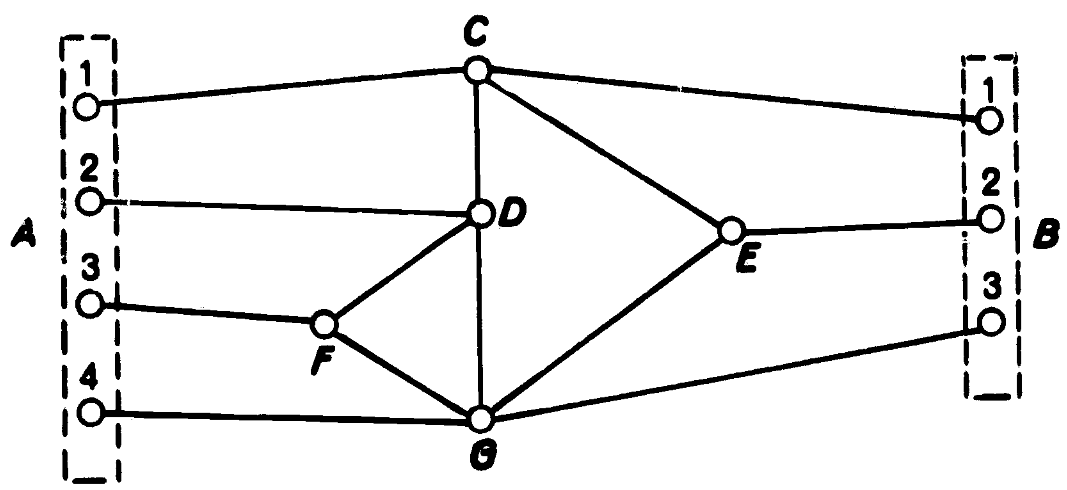
Bonus Puzzle 17: Color this "picture," including the border, using just three distinct colors so that no two faces with a common edge have the same color.



Bonus Puzzle 18: A small network of telephone lines is known to have breaks. A repairman at point *A* and another at point *B* can call each other. However, when the repairman at *A* disconnects his lines to points *C, D,* and *E* he cannot call the man at *B*. Connecting his line to point *E*, the repairman at *A* calls the man at *B* to discuss the situation. When the repairman at *B* disconnects his lines to *C* and to *F* the line goes dead until the man at *A* connects his line to point *D*. Since they now have located two breaks in the network, the repairmen decide to proceed to the broken sections to repair them. Where are the two breaks located?



Bonus Puzzle 19: A technician makes six continuity checks of a network of lines connecting two terminals. If the six checks result in the information given below, then which of the 13 segments in the network are open and which are closed?



CONTINUITY CHECKS

A–1 to B–1	Closed
A–1 to B–2	Open
A–2 to B–1	Closed
A–3 to B–1	Open
A–3 to B–2	Closed
A–4 to B–3	Closed

Glossary

Arc—set of successive edges when two vertices are connected directly or through other vertices so that no vertex is entered twice

Branches—choice of exit edges at a vertex

Branching vertex—vertex of local degree three or greater

Circuit arc—arc which returns to its initial vertex

Circuit edge—one which, with an arc, forms a circuit

Complete circuit arc—one which includes all of the vertices (not necessarily all of the edges) and returns to its initial vertex

Complete cyclic path—one which includes all the edges and returns to its initial vertex

Complete graph—one with every pair of vertices connected by an edge

Connected graph—one in which every vertex is connected to every other vertex by an arc

Contracting a graph—deleting vertices and/or edges

Cyclic path—one which returns to its initial vertex

Directed edge—one on which direction is indicated

Directed graph—one in which every edge has a one-way direction

Dual graph—a "new" graph from a given graph, with vertices the points introduced in the faces of the given graph and edges the segments drawn crossing the edges of the given graph. (For more complete explanation, see page 43.)

Edges—segments connecting vertices of a graph

Euler graph—graph with an Euler line

Euler line—complete cyclic path

Even face—one bounded by an even number of edges

Expanding a graph—introducing new vertices and/or edges

Faces—nonoverlapping regions of a polygonal graph

Graph—a geometric figure made up of certain points and line

segments connecting some or all of the points

Hamilton line—complete circuit arc

Incident—if a vertex is the end-point of an edge, the edge is incident to the vertex

Incoming edges—edges to vertex directed inward

Infinite face—region exterior to the bounding edges of a polygonal graph

Isolated vertex—one with no edge

Isomorphic graphs—those with a one-to-one correspondence between the vertices and edges

Local degree—number of edges incident to a vertex

Local incoming degree—number of incoming edges incident to a vertex

Local outgoing degree—number of outgoing edges incident to a vertex

Mixed graph—one in which some edges are directed and others are not, or in which some edges have a two-way direction

Null graph—one with isolated vertices

Odd face—one bounded by an odd number of edges

Outgoing edges — edges from vertex directed outward

Path—set of successive edges forming a route so that each edge is used just once while a vertex may be entered more than once

Planar graph—one in which the edges do not cross or have common points except at the vertices.

Polygonal graph—connected planar graph with no single edge surrounding a region

Regular of degree *r*—local degree of every vertex in a graph is *r*

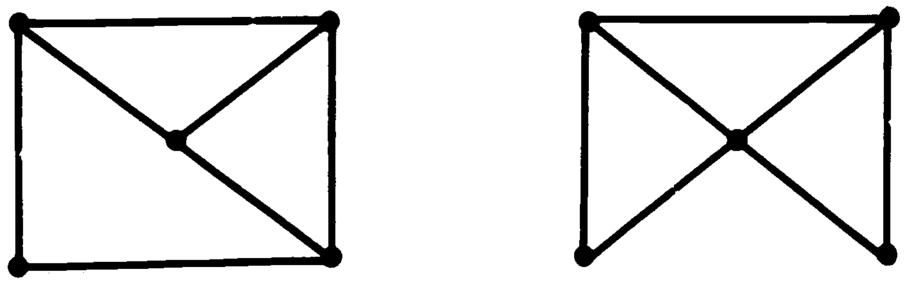
Separating edge—one which is the only connection between two vertices

Separating vertex—a vertex separating a graph into two or more parts so that every arc connecting the vertices in different parts of the graph must pass through it

Terminal edge—separating edge that separates one vertex from the remainder of the graph

Terminal vertex—one with only one edge incident to it

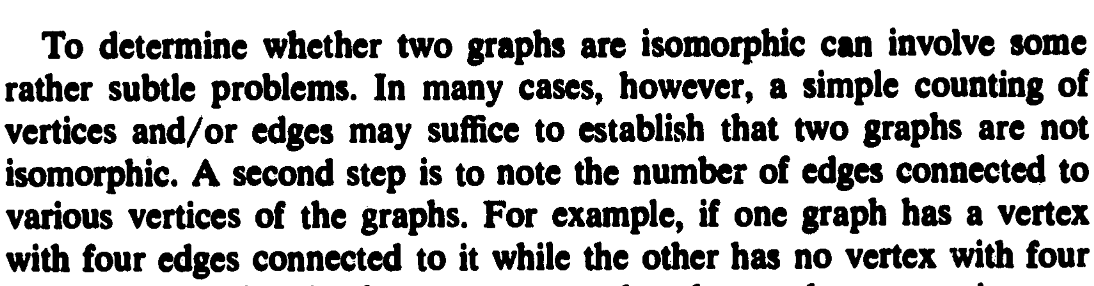
Vertices—certain points of interest on a graph



(a) (b)

Figure 14

Since our intuition is often properly led or misled by the appearance of a graph, developing an ability to construct convenient isomorphic graphs and the skill to determine when two graphs are isomorphic are worthy activities. To construct a graph isomorphic to a given graph we might begin by forming a null graph with the same number of vertices as in the given graph. Next, we can expand the null graph by introducing the required edges one by one in a convenient manner. The resulting expanded null graph should then be isomorphic to the given graph. Examining our "new" graph, we can often relocate the vertices and arrange the positioning of the edges so that our final graph is not only isomorphic to the originally given graph but also may reveal properties "hidden" in the given graph. For example, Figure 15 illustrates three such isomorphic graphs.



(a) (b) (c)

Figure 15

To determine whether two graphs are isomorphic can involve some rather subtle problems. In many cases, however, a simple counting of vertices and/or edges may suffice to establish that two graphs are not isomorphic. A second step is to note the number of edges connected to various vertices of the graphs. For example, if one graph has a vertex with four edges connected to it while the other has no vertex with four edges connected to it, then we can say that the graphs are not isomorphic. (The graphs in Figure 14 illustrate this type of situation.) After an initial examination we might then begin to identify vertices by classi-

fying them according to the number of edges connected to them. Next we can follow the "loops" around the edges in order to set up a correspondence. If we can establish a one-to-one correspondence between the vertices and edges of two graphs, then we can assert that the graphs are isomorphic. For example, the graphs in Figure 13 with their vertices identified illustrate the correspondence. The corresponding vertices of the graphs in Figure 15 have been labeled to identify the correspondence which establishes the isomorphism between them.

Now let us return to our consideration of planar graphs. Note that graphs may have edges which cross and have common points other than at the given vertices of the graph. For example, the graph in Figure 13(b) is such a graph. It has two points where the edges cross which are not vertices of the graph. It is often useful to be able to determine whether there is an isomorphic graph such that the edges do not cross or have common points other than at the vertices. For example, the graph in Figure 13(a) is isomorphic to the graph in Figure 13(b) and furthermore shows us that the graph does have the distinctive property of being connected so that the edges need not cross or have common points except at the vertices. The complete graph on four vertices shown in Figure 10(a) appears to require a crossing of the edges at a point other than at a vertex but the isomorphic graph shown in Figure 11 reveals that this is not a necessary property of the graph. That is, the complete graph on four vertices may be drawn so that there are no crossings or common points of the edges except at the vertices of the graph.

Recall that a given graph is said to be planar if an isomorphic graph can be drawn in such a way that the edges have no crossings or common points except at the vertices. Determining whether a graph is planar may be quite difficult. However, examining two particular graphs may be helpful. Consider how we might construct a complete graph on five vertices.

We begin with a null graph with five vertices as shown in Figure 8. Expand the graph by introducing the edges AB, BC, CD, DE, EA, AC, and AD. The graph thus far is planar. We must still expand the graph by introducing edges BE, CE, and BD. To avoid crossings we may introduce the edges BE and CE as shown in dashed lines in Figure 16. In order to introduce BD, however, we must cross CE or CA or AE, so that it is impossible to construct the complete graph on five vertices without crossing edges at a point other than at a vertex.

For our second particular graph, consider the puzzle of the three tenant farmers and their wells. We can connect Farm A and Farm B to each of the three wells as shown in Figure 17. Farm C can be connect-

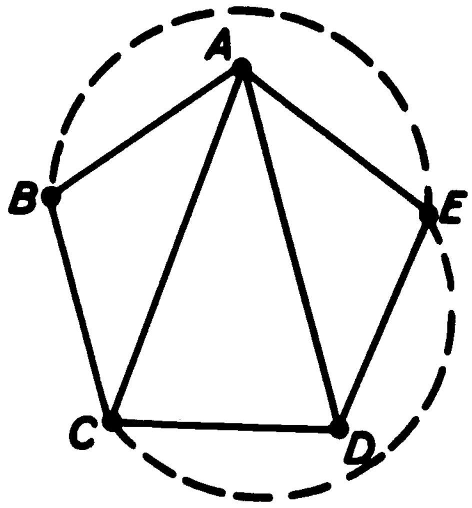


Figure 16

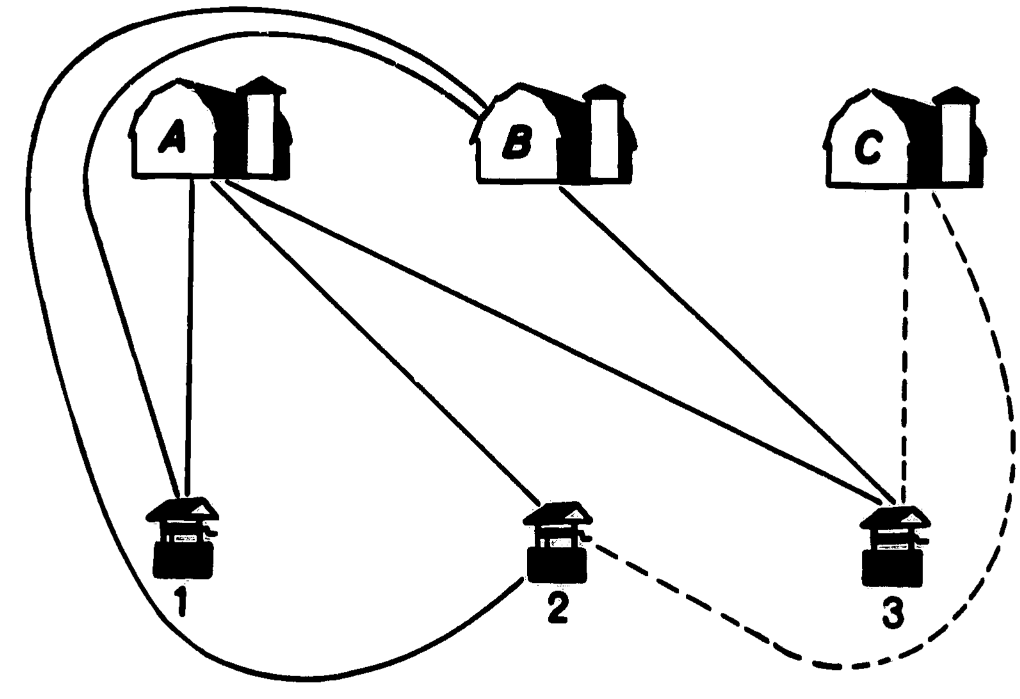


Figure 17

ed to Well 2 and Well 3 as shown by the dashed lines in Figure 17. We cannot connect Farm C to Well 1 now without crossing one of the previously constructed paths. Thus, it is impossible to construct the paths so that they do not cross. If we compare the hexagonal graph shown in Figure 12 with the graph of Figure 17 expanded by the edge from Farm *C* to Well 1, we can show that they are isomorphic. Thus the hexagonal graph in Figure 12, as well as the complete graph on five

vertices, is not planar. That is, we cannot construct isomorphic graphs for these graphs such that the edges do not cross or have common points other than at the vertices.

A Polish mathematician has shown that a graph is planar if and only if it cannot be contracted to a graph which is isomorphic to the complete graph on five vertices [Figure 10(b)] or to the hexagonal graph in Figure 12. For example, the graph in Figure 18 cannot be planar since it can be contracted to the graph shown in Figure 19 which is isomorphic to the hexagonal graph in Figure 12.

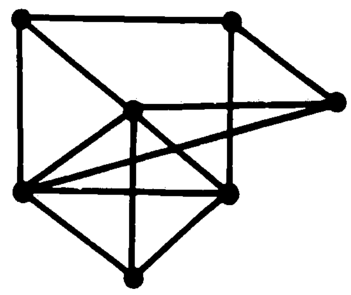


Figure 18

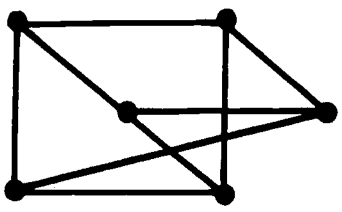


Figure 19

The graph in Figure 20, however, is planar. To see this we must construct a graph which is isomorphic to the given graph such that the constructed graph does not have edges which cross or have common points other than at the vertices. For our construction we recall that the location of the vertices can be moved and the edges need not be straight. Some experimentation might lead to the graph shown in Figure 21. Thus, since the requirements for a planar graph have been met, we can see that the graph in Figure 20 must indeed be planar.

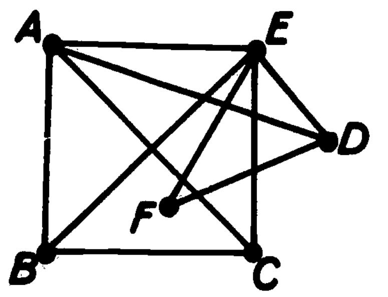


Figure 20

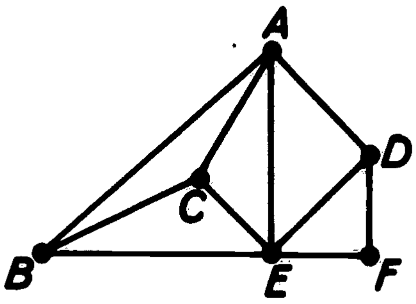


Figure 21

It can be shown that a planar graph can be drawn so that all the edges are straight, providing no pair of vertices is connected by more than one edge. This leads to planar graphs which appear like a set of adjoining polygons: for example, the graph in Figure 21. The graph in Figure 22 can be represented by the isomorphic graph in Figure 23.

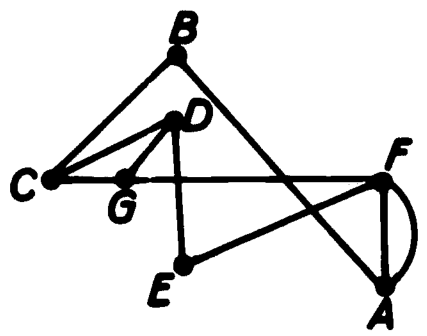


Figure 22

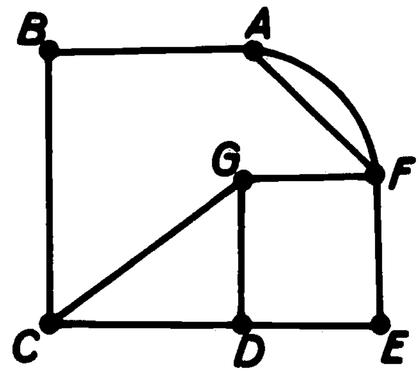


Figure 23

Figures 21 and 23 suggest a special type of planar graph. If a planar graph is connected and is such that no single edge surrounds a region, then the graph is called a **polygonal graph**. For example, the graph of the Koenigsberg Bridge puzzle shown in Figure 5 is a polygonal graph. The graph of the puzzle devised by Sir William Rowan Hamilton in Figure 2 is also a polygonal graph. To have a polygonal graph we must first have a planar graph. Furthermore, the graph must be connected; that is, we must be able to move along successive edges from any vertex to any other vertex of the graph. Finally, if no single edge "loops" around to enclose a single region, then we have a polygonal graph. The graph in Figure 24 is not polygonal since it is not planar. Figure 25 does not represent a polygonal graph even though it is planar because the graph is not connected. Although the graph in Figure 26 is planar and connected it is not polygonal, for a single edge completely surrounds a region. The graph in Figure 27 is polygonal.

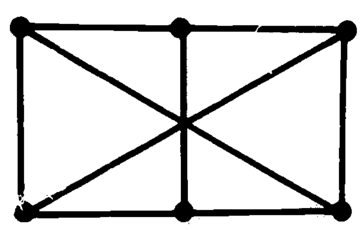


Figure 24: Not planar

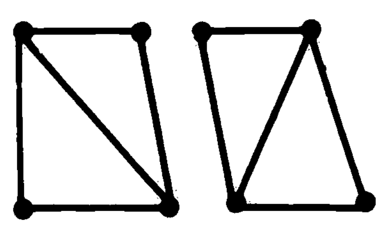


Figure 25: Not connected

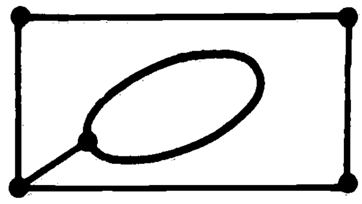


Figure 26: Surrounded

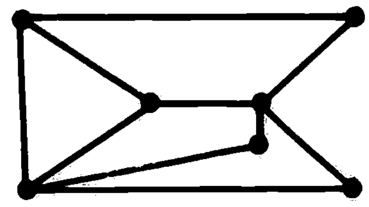


Figure 27: Polygonal

In a polygonal graph the edges and vertices bound various regions. The nonoverlapping regions are called the **faces** of the graph. It is also convenient to call the region exterior to the bounding edges of a polygonal graph a face. To distinguish this region, it may be called the **infinite face** of the graph. For example, the polygonal graph in Figure 23 has five faces. We can refer to the faces by naming the successive vertices which bound each face: *ABCGFA, CDGC, DEFGD, AFA,* and the infinite face *ABCDEFA*.

We began this section with the question, "How can we begin to understand and explain puzzles of the kind illustrated in Section 1?" We have begun by laying a general foundation for a model with the characteristics, properties, and relationships assumed to exist in the puzzles of interest. The model thus far consists of geometric figures which we call graphs, with vertices and edges. In the next section we will examine a few of the properties and characteristics of these graphs.

3 Arcs and Paths

What are the "puzzling" aspects of the puzzles we have posed? In working with the puzzles, the object was to trace a path of some special sort through the puzzle. In terms of our model graphs, this would involve the vertices and edges. For example, for Puzzle 1 illustrated in Figure 3(a) and Figure 3(b) we might ask whether there are special properties of the vertices and edges which might answer our problem and lead to a solution. For convenience in what follows, let us restrict our considerations to those graphs in which every edge connects a pair of vertices.

In our model graphs if a vertex is the endpoint of an edge, we will say that the edge is **incident** to the vertex. The number of edges incident to a vertex is called the **local degree** of the vertex. If A is a vertex, we denote the local degree of A by $d(A) = n$. For example, in Figure 28 the local degrees are

$d(A) = 4 \qquad d(B) = d(C) = d(D) = 3 \qquad d(E) = 5 \qquad d(F) = 2$

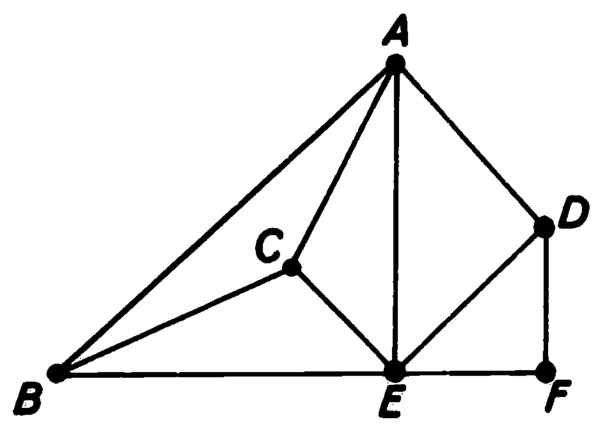


Figure 28

In a graph with edges connecting pairs of vertices each edge must have its endpoints at a vertex. Thus, the sum of the local degrees of the vertices of a graph must equal twice the number of edges in the

graph. That is, if $A_1, A_2, \ldots, A_k$ denotes the vertices of a graph and N the number of edges, then we must have

$$d(A_1) + d(A_2) + d(A_3) + \ldots + d(A_k) = 2N.$$

For example, for the graph in Figure 28, we have

$$4 + 3 + 3 + 3 + 5 + 2 = 2N, \text{ so that } 20 = 2N \text{ or } N = 10.$$

Observe that the sum of the local degrees of the vertices of a graph must equal an even number. But the local degree of a given vertex may be either even or odd. If a sum is even, however, there must be an even number of odd summands (addends). Thus, we have: *In a graph the number of vertices of odd local degree must be an even number.* For example, the graph in Figure 28 has 4 vertices of odd local degree. Our statements include those graphs with no vertices of odd local degree since 0 is considered an even number. The graph of the Koenigsberg Bridge puzzle shown in Figure 5 has all 4 vertices of odd local degree: $d(A) = d(B) = d(D) = 3$ and $d(C) = 5$.

If the local degree of every vertex in a graph is the same, say r, then the graph is said to be **regular of degree r.** For example, the complete graph on four vertices [Figure 10(a) or Figure 11] is regular of degree 3. Every complete graph of n vertices is regular of degree $n - 1$ because every vertex must have $n - 1$ edges to the other $n - 1$ vertices in the graph (see Figure 10). The hexagonal graph in Figure 12 and the graphs in Figure 13 are regular of degree 3.

We have mentioned that edges connect vertices. When two vertices are connected directly or through other vertices by edges so that no vertex is entered twice, the set of successive edges is said to form an

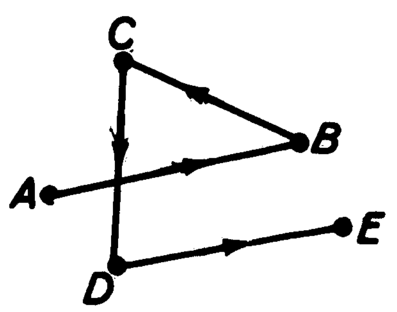


Figure 29

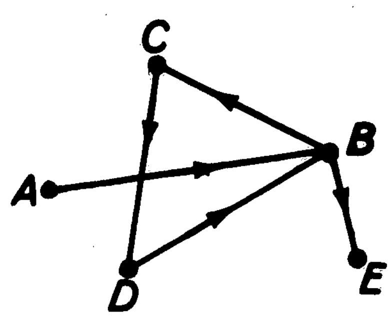


Figure 30

arc between the vertices. The graph in Figure 29 illustrates an arc. When a set of edges forms a route so that each edge is used just once while a vertex may be entered more than once, the set of successive edges is said to form a **path.** The graph in Figure 30 illustrates a path. An arc connects vertices, and in Figure 29 it can be denoted by *ABCDE* where no vertex appears twice. A path describes a "sightsee-

ing" route through the edges of a graph, where a vertex may appear twice or more and each edge just once. In Figure 30 we can denote it by *ABCDBE*. In the Koenigsberg Bridge puzzle, the object is to construct a complete path through all the edges of the graph. In Hamilton's travelers puzzle, the object is to construct a complete arc through all the vertices of the graph.

If every vertex in a graph is connected to every other vertex by an arc we say that the graph is **connected.** When a graph is connected, arcs and paths may return to their initial vertices. An arc which returns to its initial vertex is called a **circuit arc.** In a circuit arc the initial vertex is the only vertex appearing twice and edges of the graph may not have been traversed. For example, in Figure 30 the arc *BCDB* is a circuit arc. Hamilton's travelers puzzle requires us to find a **complete circuit arc** of all of the vertices of the graph. In general, any arc which returns to its initial vertex may be called a circuit arc.

A path which returns to its initial vertex is called a **cyclic path.** In a cyclic path each edge is traversed exactly once while vertices may be entered more than once. The object of Puzzle 1 in Section 2 is to find a **complete cyclic path** traversing all of the edges of the graph exactly once. The Koenigsberg Bridge puzzle can be interpreted as requiring a complete cyclic path. In general, any path which returns to its initial vertex may be called a cyclic path. Notice that every circuit arc is a cyclic path but that a cyclic path need not be a circuit arc.

The graph in Figure 31 has a complete circuit arc (as shown in Figure 36) but no complete cyclic path, whereas the graph in Figure 32 has a complete cyclic path (as shown in Figure 37) but no complete circuit arc.

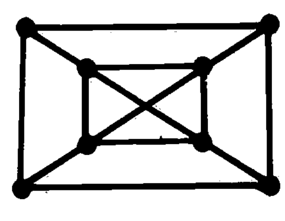


Figure 31

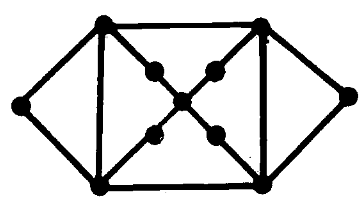


Figure 32

In considering arcs and paths we have tacitly introduced the idea of a direction along an edge. Of course, for arcs and paths, a direction on an edge is a convenience for we can usually go in either direction. However, situations may arise in which a direction is essential—for example, in considering one-way streets on a city map or a game competition in which one of two teams wins and the other loses. When a direction is indicated on an edge, we call the edge a **directed edge.** If every edge of

a graph is directed, the graph is called a **directed graph.** Figures 29 and 30 are directed graphs. We can indicate a directed edge with an arrowhead on the edge. If some edges of a graph are directed while others are not, then the graph is said to be a **mixed graph.** Figure 33 is a directed graph which depicts four teams in a pairwise competition: *A* and *B* play, and *A* wins; *C* and *D* play, and *C* wins. When *A* and *C* play, *C* is shown as winning. Figure 34 is a mixed graph, since the edges *DE* and *GF* are not directed. The directed edges indicate a path, *DCBDAB* or *DABDCB*, and a small circuit arc, *EGHE*.

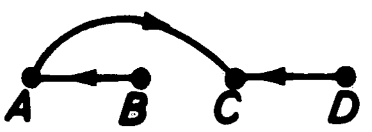


Figure 33

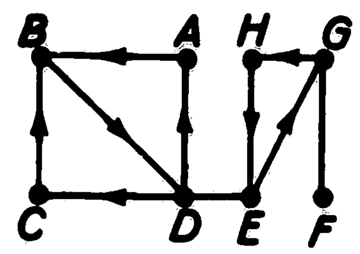


Figure 34

Under what conditions can the edges of a graph be directed so that there is a directed path from any given vertex to any other vertex in the graph? If a graph is not connected, then it is clear that there are two vertices which do not have a path connecting them. Figure 35 is such a graph. If there is a complete circuit arc, the edges of the arc can be directed in a "circular" manner so that it is possible to go from any vertex to any other vertex along the directed edges. The edges which are not traversed by the arc may be directed either way. Figure 36 shows a directed complete circuit arc. If there is a complete cyclic path, the edges of the path can be directed in a "circular" manner following the path so that each edge of the path will be directed. To go from any vertex to any other vertex we need only to follow the directed path. Figure 37 shows a directed complete cyclic path. Thus, any connected graph with a complete circuit arc or a complete cyclic path can be directed so that there is a directed path from any given vertex to any other vertex in the graph.

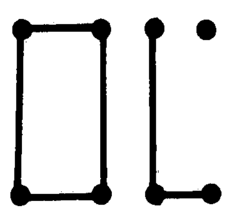


Figure 35

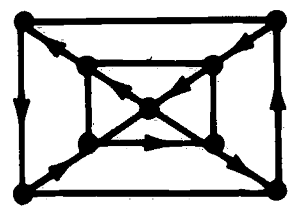


Figure 36

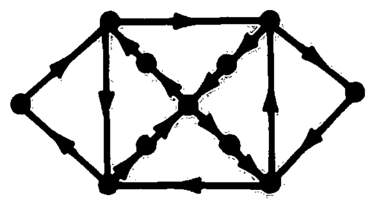


Figure 37

In studying directed paths in graphs we observe that the edges of graphs can be classified. For example, the edges *DE* and *GF* in the graph in Figure 34 are not only undirected—they are quite distinct in their relationship to the vertices as compared to the other edges of the graph. If an edge is the only connection between two vertices of a graph, it is called a **separating edge** of the graph. In Figure 38, *CD* is a separating edge of the graph. A separating edge divides a graph into two parts and is the only connection between them. If a vertex has only one edge incident to it, the edge is called a **terminal edge** and the vertex a **terminal vertex** of the graph. A terminal edge is a special type of separating edge; it separates one vertex from the remainder of the graph. In Figure 38, *EF* is a terminal edge and vertex *F* a terminal vertex of the graph. If an edge is not a separating edge, then there must be another arc connecting the two vertices to which the edge is incident. Because the arc and given edge will form a small circuit, the edge is called a **circuit edge** of the graph. In Figure 38, *AB* and *BC* are circuit edges of the graph. In Figure 34, *DE* is a separating edge and *GF* a terminal edge with terminal vertex *F*.

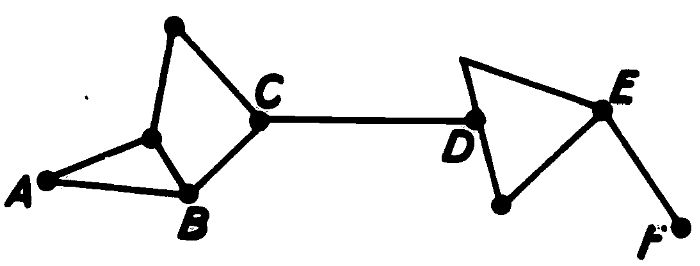
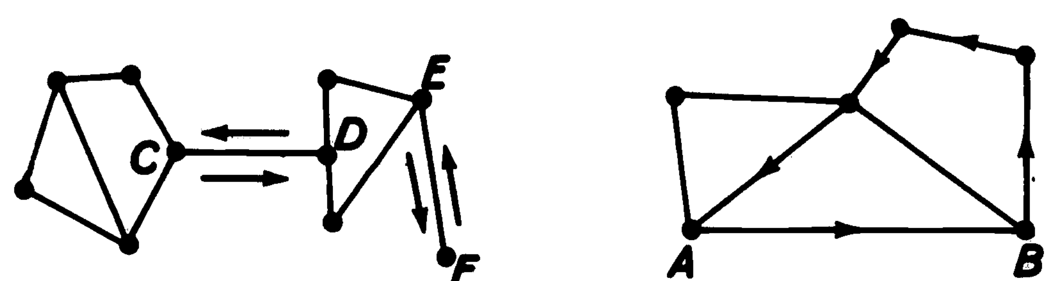


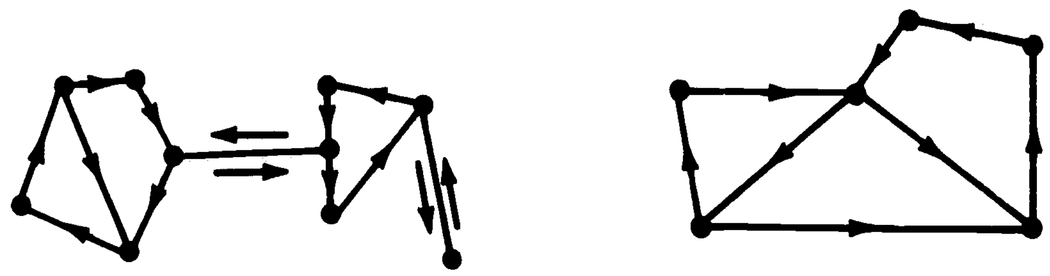
Figure 38

Recalling that a common problem in geometric puzzles is to construct paths through them, let us consider how this might be accomplished in connected graphs. We begin with two simple situations. First, a separating edge of a graph can be made into a two-way undirected edge as shown in Figure 39(a). Second, a circuit edge, because it can be made a part of a small circuit, can be directed in a "circular" manner to form a circuit of vertices as shown in Figure 39(b).



(a) Figure 39 (b)

To enlarge our considerations: If a circuit edge has a vertex in common with a separating edge, the two-way "direction" of the separating edge enables us to leave and return to the circuit to form a path covering additional vertices of the graph. If a circuit edge has a vertex in common with another circuit edge, we can direct the new circuit edge to conform to the direction of the previously directed circuit edge and thus form another directed circuit of the graph. Figures 40(a) and 40(b) illustrate these enlargements in the directioning of the edges of a graph.



(a) (b)

Figure 40

In a similar manner we can continue to enlarge a directed path in a graph until the entire graph has been "directed." That is, an undirected connected graph can always have its circuit edges directed and its separating edges made two-way so that there is a "directed" path from any given vertex to any other vertex in the graph. If there are separating edges, these edges must be made into one-way directed edges for the graph to be considered a directed graph. Otherwise, the graph must be considered a mixed graph. For example, Figure 40(a) is a mixed graph whereas Figure 40(b) is a directed graph. Figure 41 illustrates a somewhat more complicated directed graph.

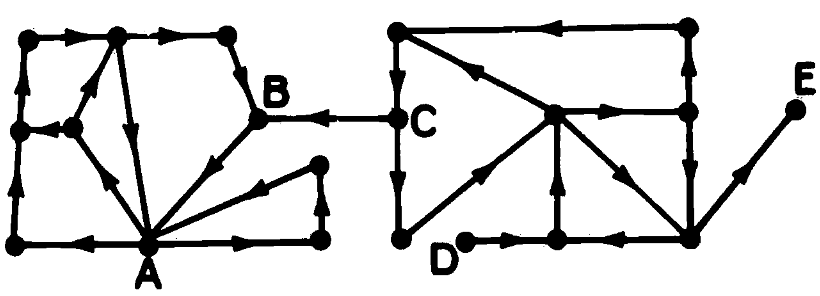


Figure 41

Notice that in the directed graph of Figure 41 we cannot go from an arbitrary vertex to many other vertices in the graph along the directed edges. The separating edge *BC* being directed from *C* to *B* shows that

once we are at a vertex to the left of B, we cannot return to the vertex C or to any vertex to the right of C. We cannot reach the terminal vertex D from any other vertex in the graph since the terminal edge incident to D is directed away from D. Once we enter the terminal vertex E we cannot leave because the terminal edge incident to E is directed to E. We might ask, "Is there a path from some vertex through the directed edges which will take us through all the other vertices of the graph?" If there is, we must begin at the vertex D and end at the vertex E. But, to reach all the vertices, we must traverse the separating edge BC. If we traverse the edge from C to B, we cannot return to end at E. Thus, there can be no path in the graph from any vertex which will take us through all other vertices of the graph along the directed edges.

If a connected graph has separating edges, then there can be no complete circuit arc or cyclic path for the graph. This is evident since a separating edge would require a two-way direction and the use of a vertex twice. Recall that a path allows just a single use of an edge and an arc only one entry and exit from a vertex.

Now let us return again to a consideration of the number of edges incident to a vertex. In a directed graph we can describe the edges incident to a vertex as **outgoing edges** from the vertex or as **incoming edges** to the vertex. Thus, we can distinguish between the **local outgoing degree** and the **local incoming degree** of a vertex in a directed graph. If A is a vertex, we can denote the local outgoing degree of A by $d_o(A) = n$ and the local incoming degree of A by $d_i(A) = m$. Notice that $d(A) = d_o(A) + d_i(A)$. For example, Figure 42(a) shows a vertex A with $d_o(A) = 3$, $d_i(A) = 2$, and $d(A) = 5$. Figure 42(b) illustrates a vertex B with $d_o(B) = 4$ and $d_i(B) = 0$, so that $d(B) = d_o(B)$. Figure 42(c) shows a vertex C with $d_o(C) = 0$ and $d_i(C) = 3$, so that $d(C) = d_i(C)$.

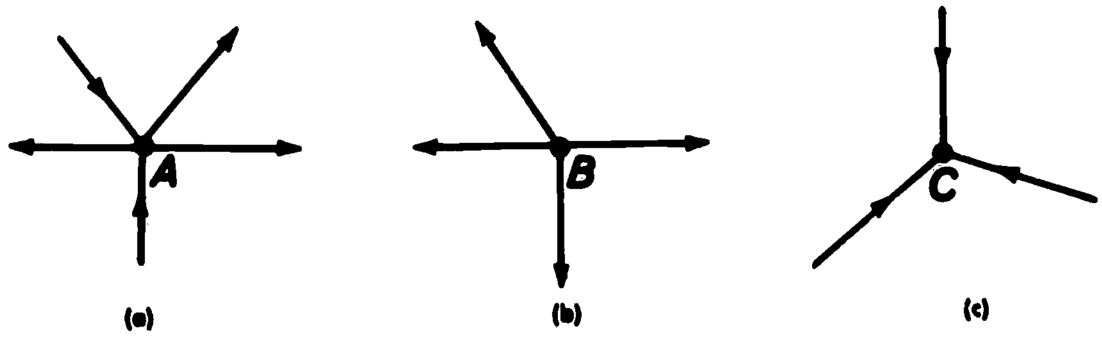


Figure 42

Because a directed edge has one initial and one terminal end, we can obtain the number of edges in a directed graph either by summing the

local outgoing degrees of all the vertices or by summing the local incoming degrees of the vertices. For example, consider the directed graph shown in Figure 43. In Table 1 we have tabulated the local incoming, the local outgoing, and the local degree of each vertex. The number of edges in the graph is given by the sums (totals) of the local incoming and local outgoing degrees, and is equal to one-half the sum of the local degrees of the vertices.

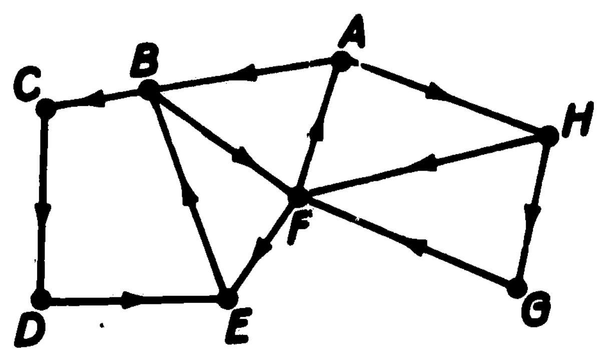


Figure 43

TABLE 1

VERTEX	Local Incoming Degree	Local Outgoing Degree	Local Degree
A	1	2	3
B	2	2	4
C	1	1	2
D	1	1	2
E	2	1	3
F	3	2	5
G	1	1	2
H	1	2	3
Total	12	12	24

If the local degree of a vertex of a directed graph is even, then the local outgoing and local incoming degrees of the vertex must be both even or both odd. If the local degree of a vertex of a directed graph is odd, then the local outgoing or local incoming degree of the vertex (but not both) must be odd. These observations can be verified in Figure 43 and Table 1. They are illustrated in Figures 44(a) and 44(b).

(a) (b)

Figure 44

Since the initial and terminal ends of a complete cyclic path must be the same vertex and each edge is traversed just once, it is evident that a complete cyclic path cannot be drawn in a connected graph unless every vertex of the graph is of even local degree. Consider also that a path which enters a vertex must leave the vertex unless the vertex is the terminal end of the path. Now, recalling the Koenigsberg Bridge puzzle, we can say that the promenade is impossible because its graph has vertices of odd local degree.

In this section we have developed and discussed some of the characteristics and properties expected in graphs representing geometric puzzles. In the next section we will examine the specific problems posed by Euler's Koenigsberg Bridge and Hamilton's travelers puzzles.

4 Euler and Hamilton Lines

In what graphs is it possible to find a complete cyclic path? Because of Euler's article on graphs which poses this question, such a cyclic path is called an **Euler line** and a graph with an Euler line is called an **Euler graph.** Our conclusion in the last section was that the Koenigsberg Bridge puzzle did not represent a graph with an Euler line. Figure 45 shows an Euler graph with an Euler line drawn on it. Any vertex may be taken as the starting and finishing point.

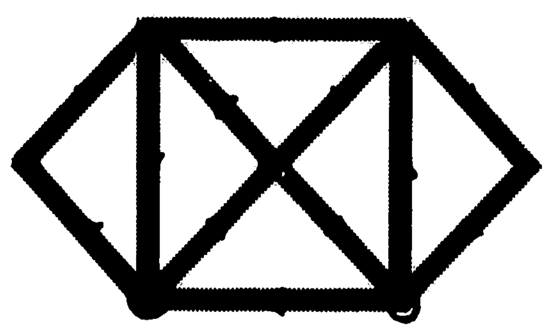


Figure 45

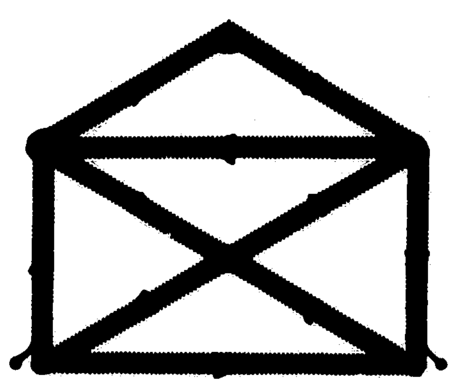


Figure 46

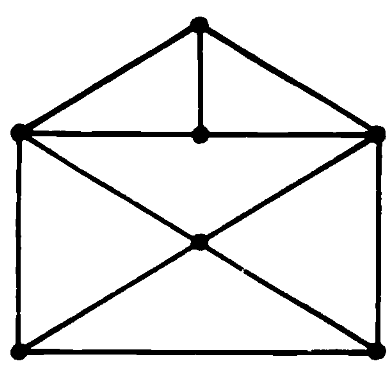


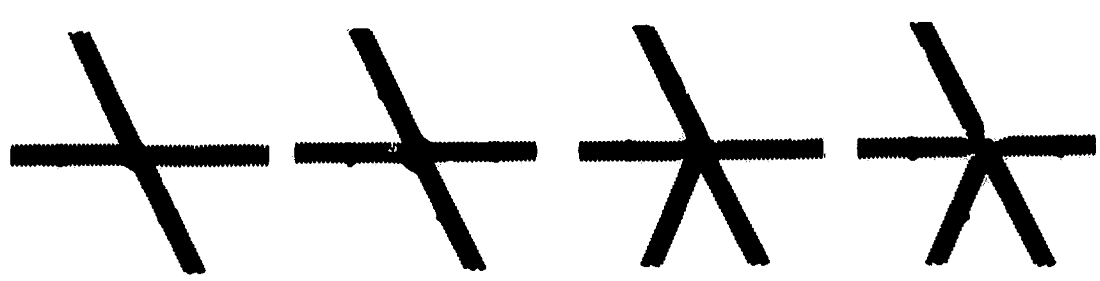
Figure 47

Puzzle 1, depicted with Figures 3(a) and 3(b) on page 3, asked you to trace the figures without taking the pencil from the paper. Figure 46

depicts the solution for Figure 3(a): It does not have an Euler line, but it does have a complete (not cyclic) path. Figure 3(b), whose graph is shown in Figure 47, does not have an Euler line nor even a complete path.

Careful examination of the graphs in Figures 45, 46, and 47 will show that the graphs differ in the local degree of certain of their vertices. For example, every vertex in the graph in Figure 45 is of even local degree. The terminal and initial vertices of the complete path in Figure 46 are of odd local degree while all of the remaining vertices in the graph are of even local degree. The graph in Figure 47 has four vertices of odd local degree.

Imagine drawing a path on a graph. As we draw the path we can direct the edges and "delete" them from further consideration. If we enter a vertex of even local degree, we can always leave the vertex, "deleting" two edges at a time until every edge has been used just once. If we enter a vertex of odd local degree, we have an even number of edges remaining incident to the vertex so that on leaving there is an odd number of edges left incident to the vertex. Now, eventually the vertex of odd local degree will have one edge remaining and upon entering the vertex, we will be unable to leave. Thus, a vertex of odd local degree must be a terminal end of a path on a graph, either the beginning or the end. Since a complete cyclic path has no terminal ends and uses all the edges of a graph, there can be no complete cyclic path in any graph with a vertex of odd local degree. Also, because a complete path has just two ends and uses each edge once, there can be no complete path in a graph with more than two vertices of odd local degree. Figure 48(a) and Figure 48(b) illustrate our imagined entry and exit from vertices of even and odd local degrees.



(a) (b)

Figure 48

Any connected graph with even local degree at every vertex has an Euler line.

To construct an Euler line on an Euler graph we can begin at any convenient vertex. We draw a directed path from vertex to vertex tra-

versing undirected edges and directing them. Because the local degree of each vertex is even we can always leave the vertex unless we return to the initial vertex. If our directed path has traversed all edges, we have constructed an Euler line. If there are undirected edges, there will be a vertex on our directed path to which an undirected edge is incident. But, since the local degree of this vertex was even, there must be an even number of undirected edges incident to the vertex. This must be true at every vertex with an undirected edge incident to it. Figure 49 illustrates the beginning of an Euler line in an Euler graph.

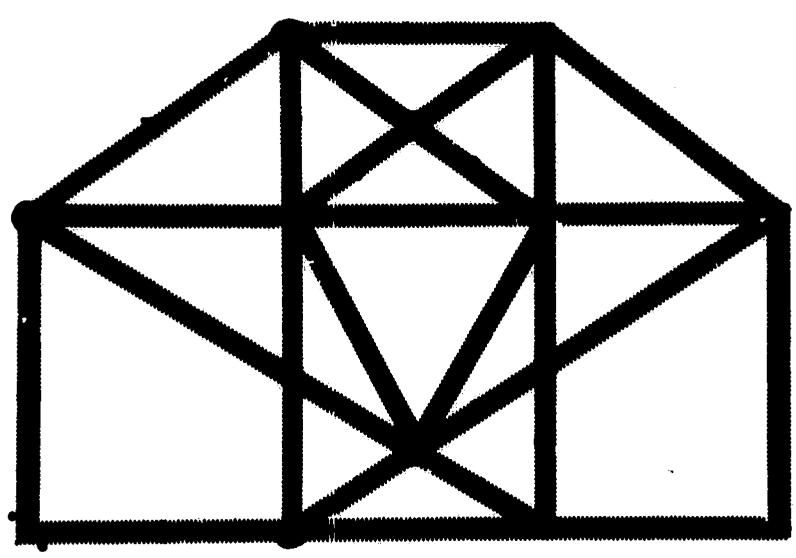


Figure 49

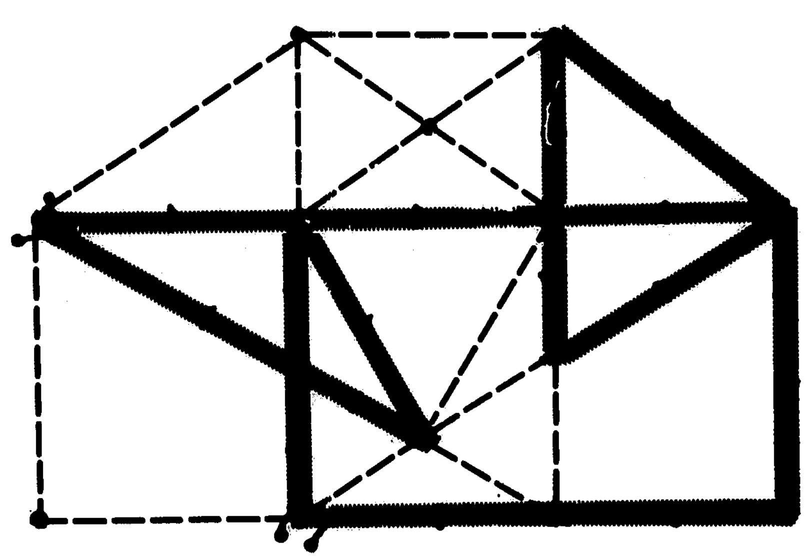


Figure 50

We can expand on our directed path from any vertex with an undirected edge. We proceed to direct these remaining edges as before, tra-

versing only those edges which are not yet directed and directing them. As before, we can always leave a vertex we enter unless we return to our initial vertex. Eventually we must do so and have thus expanded our directed path. If now our directed path has traversed all the edges, we have constructed an Euler line. If not, we can repeat the process and expand the path still further until we have attained an Euler line. Figure 50 shows the graph of Figure 49 with the previously directed edges in dashed lines, and the expansion of our directed path.

We can, if we wish, "smooth" the path constructed above to avoid "cross-overs" at vertices by rearranging the connections between incoming and outgoing edges. The final complete cyclic path or Euler line for the graph in Figure 49 is shown on the graph in Figure 51. Notice that the direction has been omitted from the path. The path could begin and end at any vertex, and could go in either direction.

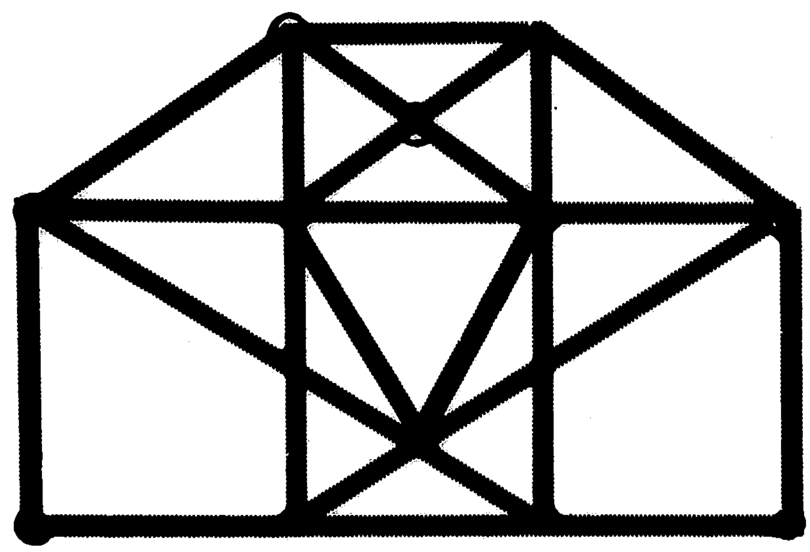


Figure 51

Euler lines impose strong restrictions on a graph. The requirement that the local degree of every vertex be even limits the edges of Euler graphs to circuit edges. To see this, we note that a separating edge divides a graph into two parts and forms the only connection between them. Thus a path through a separating edge cannot be cyclic. Furthermore, a separating edge is incident to a vertex of even or odd local degree. If the vertex is of even local degree, there must be an odd number of edges other than the separating edge incident to the vertex. Thus, there must be a vertex of odd local degree connected to this vertex through edges other than the separating edge. We can conclude that if a connected graph has a separating edge, then it must have at least two vertices of odd local degree; one on each "side" of the sepa-

rating edge of the graph. Figure 52 is a connected graph with a separating edge incident to one vertex of even local degree and one vertex of odd local degree.

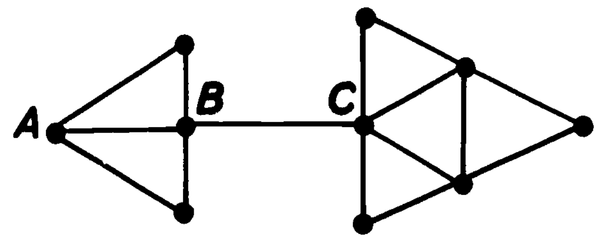


Figure 52

In many situations we can drop the cyclic requirement for a complete path through a graph. That is, we can ask in what graphs it is possible to find a complete path traversing all edges. The initial and terminal ends of the path may be at different vertices.

In order to have a complete path a graph must be connected. If a path has its initial end at a vertex of even local degree, it must terminate in the same vertex. If a path has its initial end at a vertex of odd local degree, it must terminate at some other vertex. Furthermore, if a graph has one vertex of odd local degree, it must have a second vertex of odd local degree. Because each vertex of odd local degree must be a terminal end of a path, we have: *Any connected graph with exactly two vertices of odd local degree will have a complete path traversing all edges of the graph. The initial and terminal ends of the path will be the vertices of odd local degree.*

The graph in Figure 52 has exactly two vertices, A and C, of odd local degree. A complete path for the graph with terminal ends at A and at C is illustrated in Figure 53.

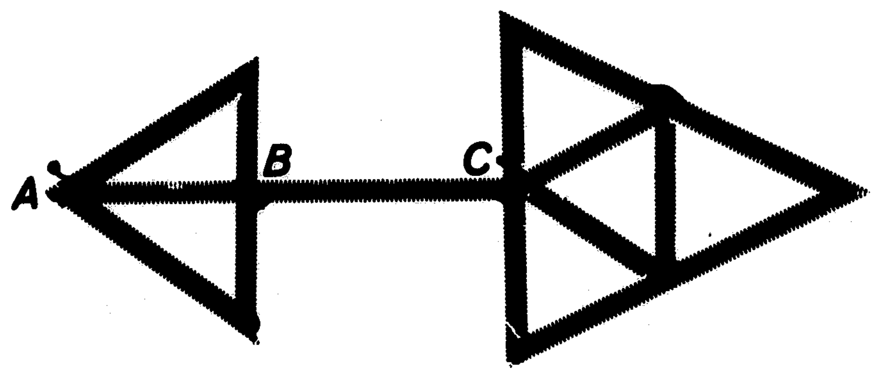


Figure 53

If a connected graph with two vertices of odd local degree is expanded by connecting the vertices of odd local degree with an edge, the

resulting connected graph will have all vertices with even local degrees and thus an Euler line.

Since a connected graph will have an even number of vertices with odd local degrees, we can generalize our results to the number of paths necessary to traverse all edges of a graph. Any connected graph with $2K$ vertices of odd local degree will require K paths which, taken together, will traverse all edges of the graph exactly once. The graph can be expanded to an Euler graph with the addition of exactly K edges connecting the $2K$ vertices of odd local degree.

Figure 54 illustrates a connected graph with two vertices of odd local degree. A complete path can be drawn on the graph. The addition of the edge shown in dashed lines would result in the graph becoming an Euler graph. Figure 55 shows a graph with six vertices of odd local degree. Thus, three paths would be required to traverse all of the edges exactly once. Addition of the three edges shown in dashed lines would result in an expanded graph that would be an Euler graph.

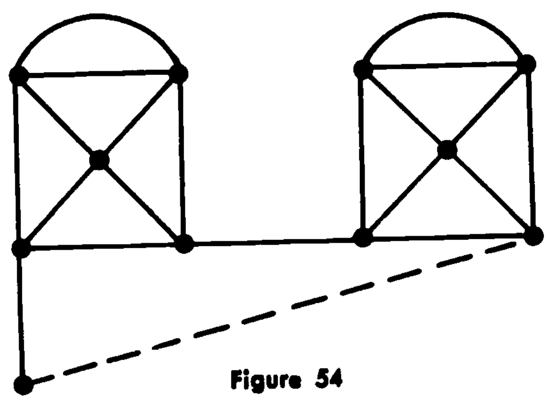


Figure 54

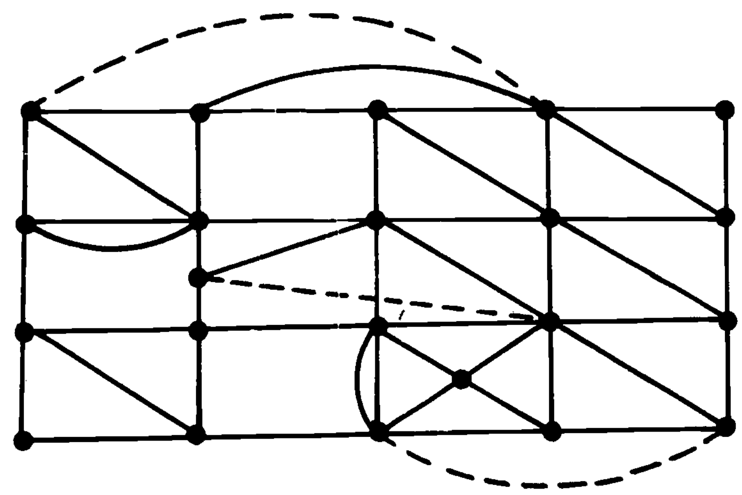


Figure 55

Given a graph, when is it possible to find a complete circuit arc which passes through each vertex of the graph exactly once? Recall that

Hamilton's travelers puzzle requires us to find such a complete circuit arc of a graph. When a complete circuit arc can be found for a graph it is called a **Hamilton line.** Thus, a Hamilton line is a succession of edges which enter each vertex of a graph exactly once to form a "route" through the vertices of the graph and such that no vertex except the initial vertex appears twice.

A Hamilton line does not necessarily traverse all the edges of a graph. As a matter of fact, since an arc may enter a vertex just once, a Hamilton line traverses exactly two edges incident to each vertex of the graph, once to enter the vertex and once to leave the vertex. Because a Hamilton line must pass through each vertex of a graph, it is clear that the graph must be connected. Furthermore, since a circuit is necessary through all of the vertices of the graph, we cannot have separating edges in the graph. Thus, in order to have a Hamilton line, all of the edges in a graph must be circuit edges. Figure 56 shows a Hamilton line drawn on the graph representing Hamilton's travelers puzzle.

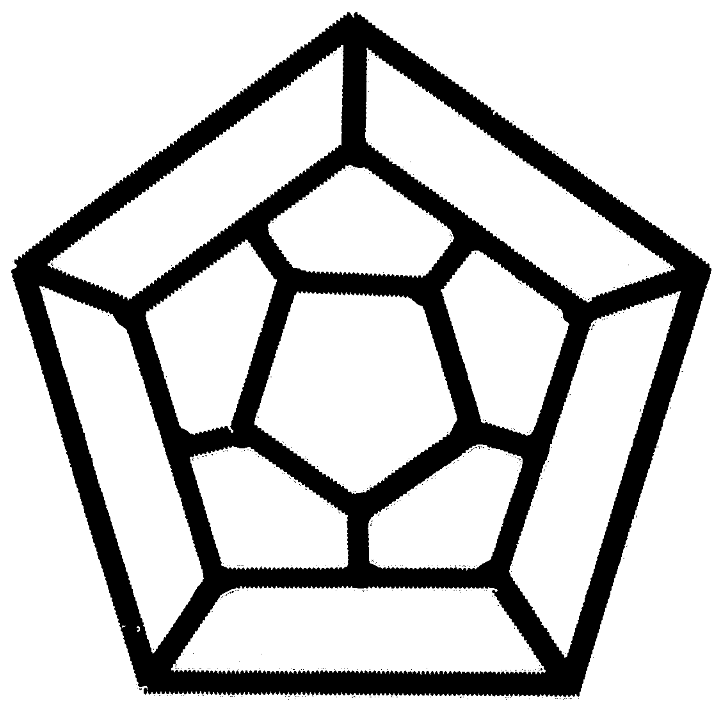


Figure 56

Although the Euler line and Hamilton line problems appear to have much in common, they are subtly different. Recall the graphs shown in Figures 31 and 32. Figure 31 has a Hamilton line, as shown in Figure 36, but no Euler line. Figure 32 has an Euler line, shown in Figure 37, but no Hamilton line.

To examine a graph for an Euler line it was sufficient to note the local degrees of the vertices of the graph. For a Hamilton line, however, we must examine the succession of edges forming an arc. Once a vertex has been entered with an arc, we cannot re-enter the vertex. For example, in Figure 57, if an arc enters either vertex *B* or vertex *D*, both the remaining edges must be traversed in order to pass all of the vertices of the graph. Once we have traversed one of the edges incident to a vertex, we cannot return to the vertex through any other edge for the vertex can be entered only once. Thus, the graph in Figure 57 cannot have a Hamilton line.

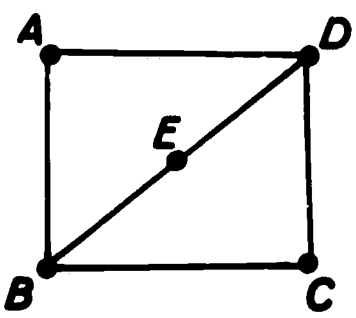


Figure 57

Consider how an arc might be constructed in a graph. We begin at some vertex in the graph and pass through successive vertices. At each vertex entered, we have either a single exit edge or a choice of exit edges to take in continuing the arc. If there is a single exit edge, we must use this edge in order to continue the arc. If there are two or more exit edges to take in continuing the arc, we must choose one of the edges to continue the arc. In doing so the remaining edges incident to the vertex become useless in a continuation of the arc. For example, in

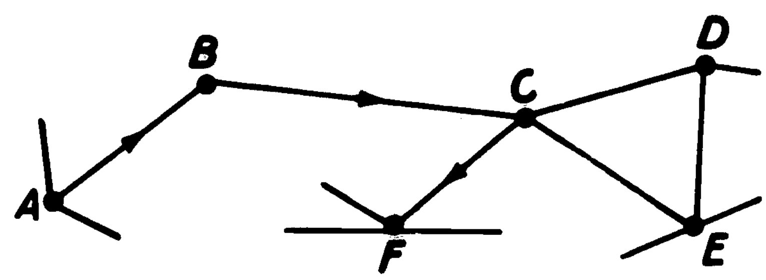


Figure 58

the partial graph in Figure 58, we begin an arc at vertex *A* and traverse the edge *AB* to *B*. At *B* we have no choice but to continue the arc to vertex *C*. At vertex *C*, however, we have a choice of *CD*, *CE*, or *CF* as exit edges to continue the arc. If we choose the edge *CF* to continue the arc to vertex *F*, the edges *CD* and *CE* become useless in a continuation of the arc. When we have such a choice of exit edges at a vertex to

continue an arc, we say that the arc has **branches** at the vertex. In Figure 58, the arc has branches at vertex *C*, vertex *D*, and vertices *E* and *F*. We also say that a vertex of local degree three or greater is a **branching vertex** of a graph. That is, in Figure 58, the vertices *A*, *C*, *D*, *E*, and *F* are shown as branching vertices.

Now let us consider how we can construct a Hamilton line for the graph shown in Figure 59. We can begin at some convenient vertex, say *A*. We have three choices of edges: *AB*, *AD*, or *AL*. If we were to choose *AD*, we would have to traverse at least two more edges incident to the vertex *D*: *DC* and *DE*, *DF*, or *DH*. Thus, we avoid this edge and traverse the edge *AB* (or *AL*). From the vertex *B* we have no choice except to proceed to vertex *C* and thence to vertex *D*. At vertex *D* we have three choices of exit edges: *DH*, *DF*, or *DE*. If we choose the edge *DH*, then we have three choices at *H*. But the arc must pass through all the vertices,and each of these choices at *H* would "cut off" vertices. Similarly, if we choose the exit edge *DF* from vertex *D*, we would have to proceed to *E* which would "cut" the remaining vertices off or would have to proceed to *G* leaving vertex *E* isolated. Thus, we choose the edge *DE* to continue our arc. Figure 60 shows our progress with the "used" edges shown in dashed segments. We can denote our arc thus far by *ABCDE*. As we pass through vertex *D* using the edges *CD* and *DE*, the edges *AD*, *DH*, and *DF* become "useless" for continuing the arc, since with an arc we can enter a vertex only once. *AD*, *DH*, and *DF* are, therefore, also shown in dashed segments.

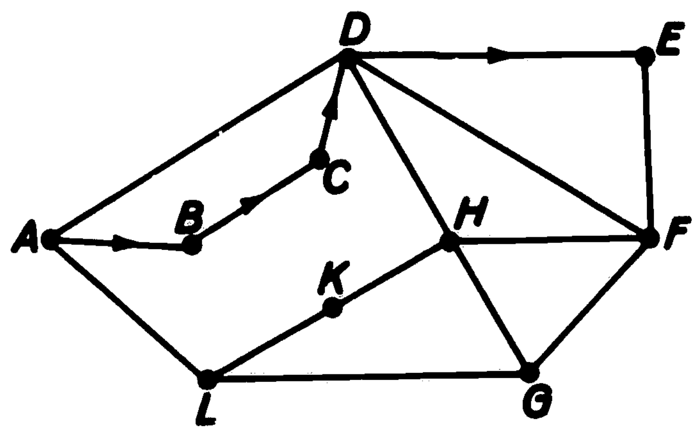


Figure 59

From vertex *E* we have no choice but to continue our arc to *F*. At vertex *F* we have two choices: *FH* or *FG*. If we traverse *FH*, then we are faced with having to traverse both *HG* and *HK* in order to pass both vertices *G* and *K*. Thus, we traverse *FG* to *G* from whence we must proceed to *H* in order to pass it on our arc. From *H* we have no

choice except to proceed to *K*, then to *L*, and, finally, our arc completes its circuit to vertex *A*. The complete Hamilton line is: *ABCDEFGHKLA*, as shown in Figure 61.

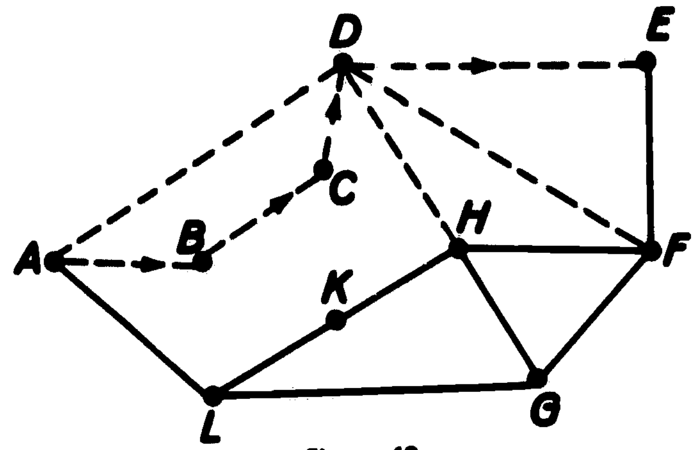


Figure 60

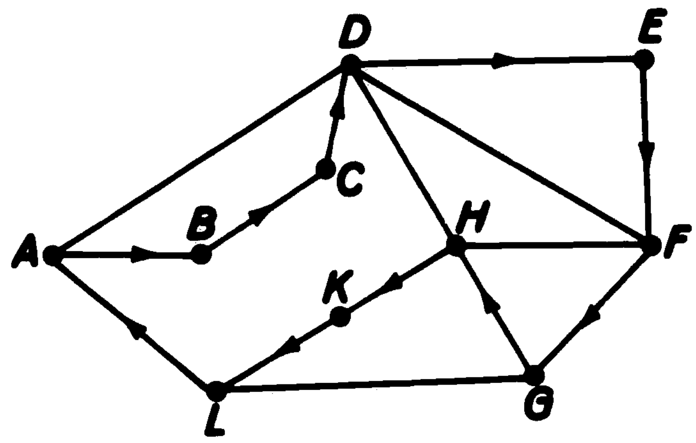


Figure 61

For a second example, consider the graph in Figure 62. We might begin an arc at vertex *A* and proceed to vertex *B*, then to vertex *C* and *D* to form *ABCD*. At vertex *D* we have two choices for continuing the arc. If we continue the arc through *DE*, the edge *DN* can no longer be traversed, for then we would be re-entering the vertex *D*. If we continue the arc through *DN*, the edge *DE* becomes useless for a continuation of the arc. In continuing the arc through the branching vertex *D* to *N*, all the edges incident to the vertex are eliminated with respect to the arc. If the continuation of the arc results in a terminal or isolated vertex with respect to the arc except for the endpoints of the arc, then the arc cannot be continued to form a complete circuit arc for the graph because a vertex will be omitted from the circuit. That is, the arc would have to have two terminal ends, the initial vertex and the terminal vertex, or leave the isolated vertex.

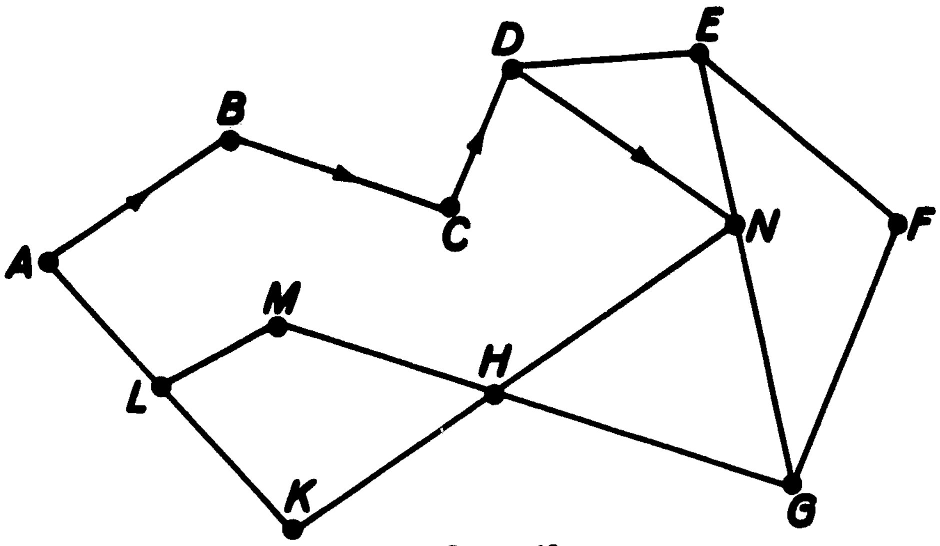


Figure 62

If no terminal or isolated vertex with respect to the arc is formed, then we continue the arc by entering and exiting from vertices not yet passed. Figure 63 illustrates this procedure for the graph in Figure 62 with the arc *ABCDNEFGH*. The edges traversed by the arc and the edges useless with respect to the arc are shown in dashed segments. At vertex *H* in the graph we find that both possible branches for our arc will result in leaving a terminal edge and vertex in the graph, other than the ends of the arc, so that we conclude that our arc cannot form a Hamilton line in the graph.

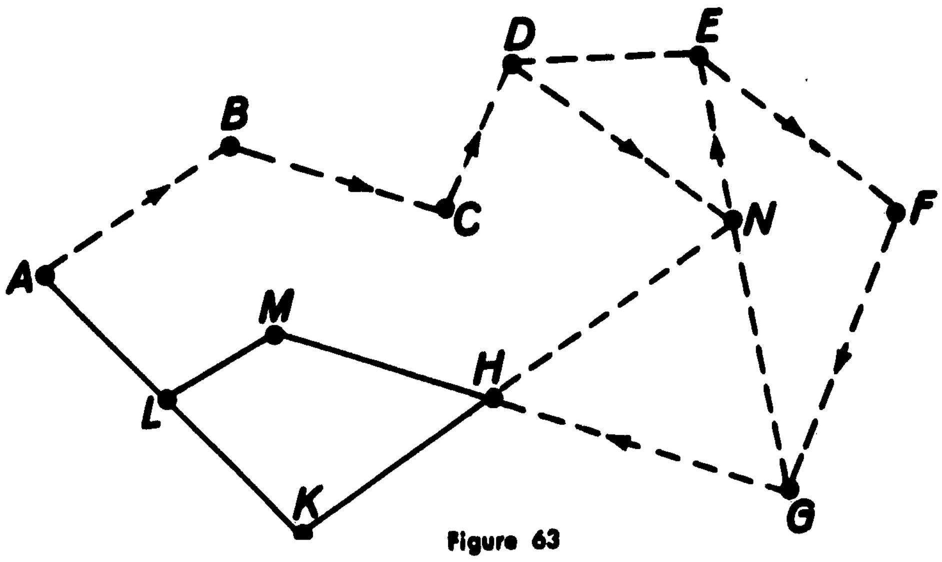


Figure 63

Our construction of an arc is not a "positive" construction which will determine the existence of a complete circuit arc. Rather, it will indicate only when a given arc cannot be continued to form a complete circuit arc. Consider the graph in Figure 64. Suppose we begin an arc at vertex

A and traverse *AB* to *B*. At *B* we have three choices for continuing the arc: *BC, BH,* or *BG*. Each of these choices results in a "loss" of two edges but leaves all of the remaining edges as circuit edges.

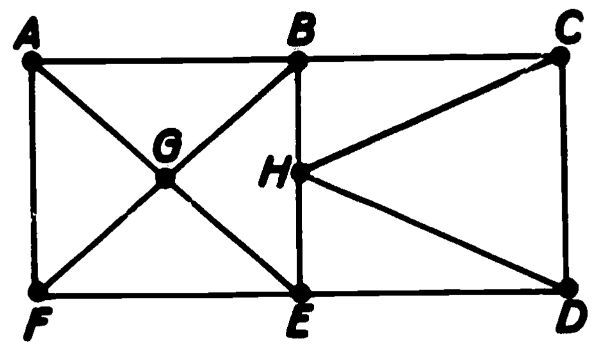


Figure 64

Suppose we continue the arc to vertex *C*. Upon "deletion" of used edges, our graph would appear as shown in Figure 65. From vertex *C* we could continue our arc to *H* or to *D*, in either case "losing" an edge but leaving the remaining edges circuit edges. If we continue our arc to vertex *D*, we must proceed to *H*, for otherwise continuing to *E* would make *H* become a terminal vertex. If we continue our arc to vertex *H*, we must proceed to *D*, for otherwise continuing to *E* would make *D* become a terminal vertex. Figure 66 illustrates the results with the arc *ABCDH*. From *H* the only possibility is to continue to *E*, where we have two choices: *EG* or *EF*. If we continue the arc to *G* we must proceed to vertex *F* and then to our initial vertex *A* to complete our Hamilton line. If we continue the arc to vertex *F* we must next include vertex *G* and then traverse *GA* to complete our Hamilton line. In either case, we have completed the desired Hamilton line.

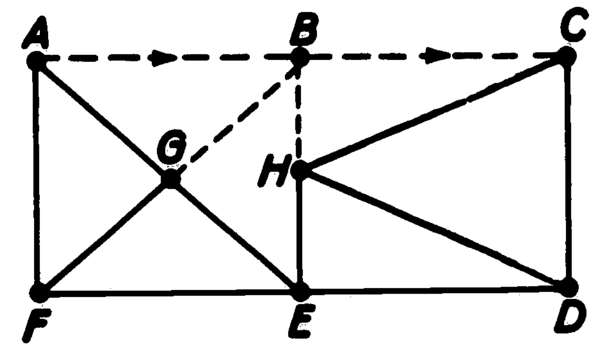


Figure 65

Now returning to the beginnings of our arc, suppose we had elected to continue the arc from vertex *B* to *G* to form *ABG*. Upon "deletion" of used edges, our graph would have appeared as shown in Figure 67.

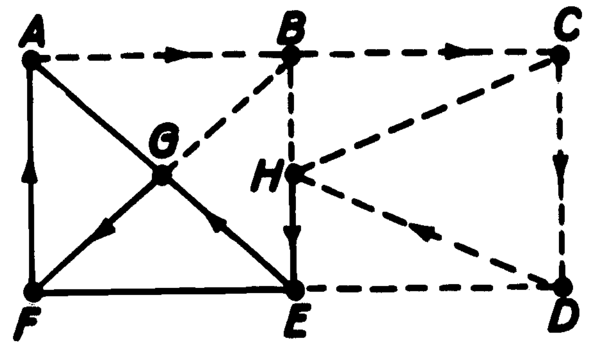


Figure 66

Although the remaining edges are all circuit edges, notice that vertex E "separates" the remaining graph into two parts. Both the initial and terminal ends of our arc ABG are on the same side of the graph, separated by the vertex E. Thus, if we continue the arc through E to pass vertices C, D, or H, we could not complete the circuit because this would require re-entering the vertex E.

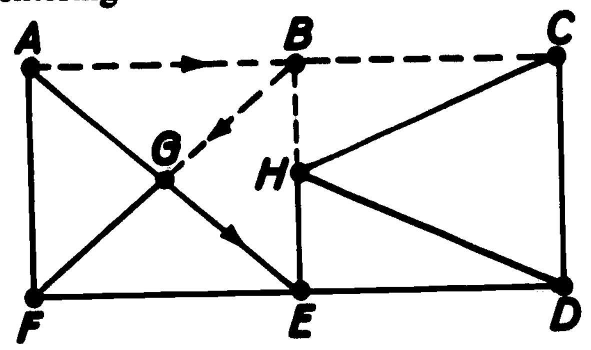


Figure 67

If a vertex separates a graph into two or more parts such that every arc connecting the vertices in different parts of the graph must pass through this vertex, then the vertex is called a **separating vertex** of the graph. For example, vertex C in Figure 68 is a separating vertex of the graph.

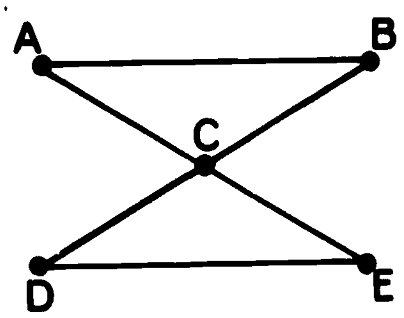


Figure 68

If a graph has a separating vertex, then an arc must pass through the vertex to connect vertices in the separated parts of the graph. Once an arc has passed through a separating vertex, it cannot return to its initial

end to complete a circuit arc because it would have to pass through the separating vertex a second time. Thus, any graph with a separating vertex cannot have a Hamilton line (a complete circuit arc).

Furthermore, in constructing an arc in a given graph, if at any stage in the continuation of the arc a separating vertex occurs, so that the initial and terminal ends of the arc are on the same side of the separating vertex, then the arc cannot be continued to form a Hamilton line for the graph.

Returning to our observation of the graph in Figure 57 and to our conclusion concerning the construction of a Hamilton line for the graph in Figure 62, notice that we entered a branching vertex with exit edges leading to vertices which did not branch. Continuing the arc would have led to a terminal edge and vertex other than at the ends of the arc. If a graph has a branching vertex such that on entering the vertex from any edge there are more than two exit edges incident to nonbranching vertices, then a complete circuit arc cannot be constructed for the graph.

Although we have not completely answered the question of when it is possible to construct a Hamilton line in a given graph, we have made some useful observations indicating when such a line could not be constructed.

If a graph is not connected or has a separating edge or separating vertex, a Hamilton line cannot be constructed on the graph.

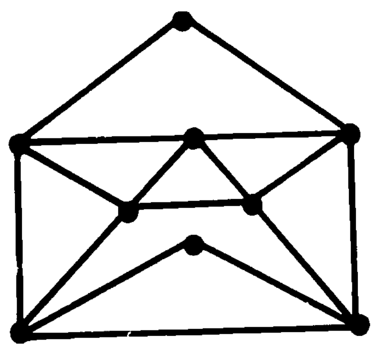
If a graph has a branching vertex with more than two exit edges incident to non-branching vertices, then a Hamilton line cannot be constructed on the graph.

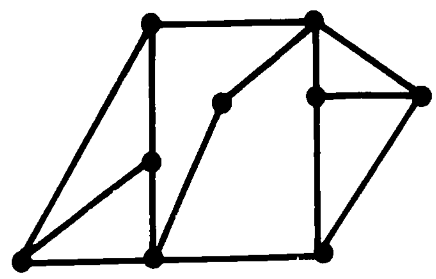
Even if a given graph is connected, has no separating edges or separating vertices, and no branching vertices with more than two exit edges incident to nonbranching vertices, we still cannot assert that there is a Hamilton line for the graph. For these graphs there appears to be no simple approach. As a matter of fact, no general solution is known for the question. Perhaps none exists and the only way of determining whether a Hamilton line can be constructed may be simply a matter of trial and error in these cases.

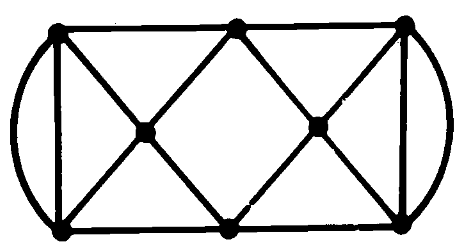
In this section we have concentrated on paths and arcs forming Euler and Hamilton lines. An Euler line is a complete cyclic path of the kind required in the Koenigsberg Bridge puzzle. A Hamilton line is a complete circuit arc of the kind required in Hamilton's travelers puzzle. For your enjoyment, we conclude with the following puzzle graphs.

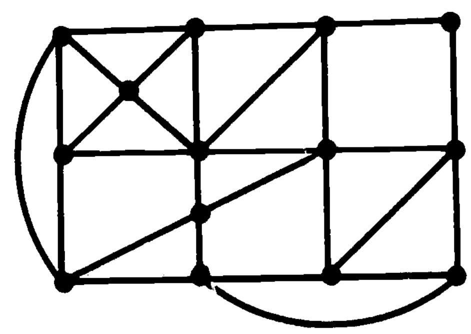
Determine whether an Euler line can be drawn in each graph. If it can, draw it. If it cannot, can a complete path be drawn?

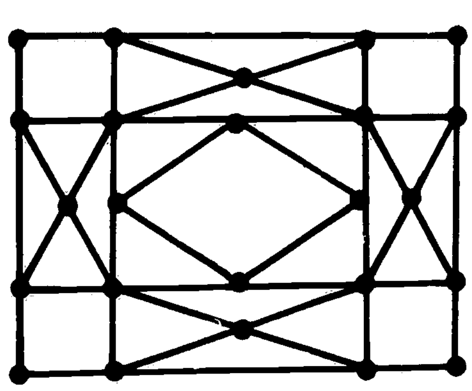
Determine whether a Hamilton line can be drawn in each graph. If it can, draw it. If it cannot, can a complete arc be drawn?

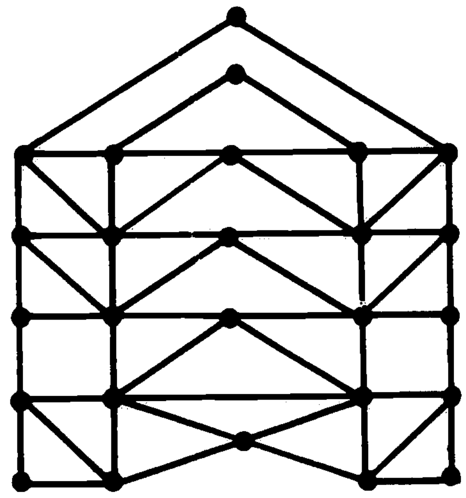


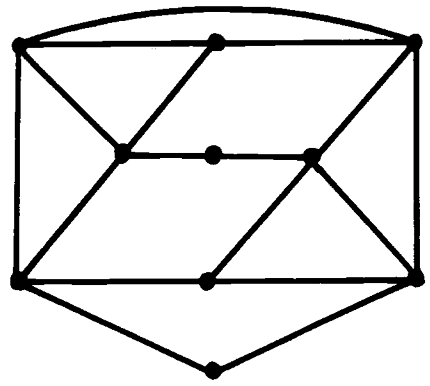


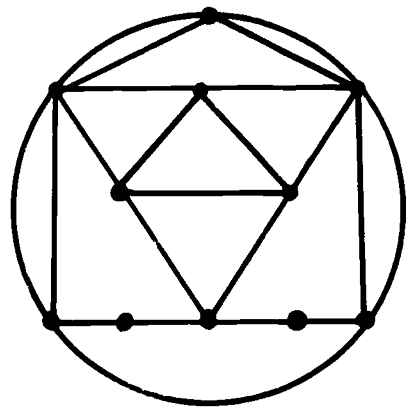


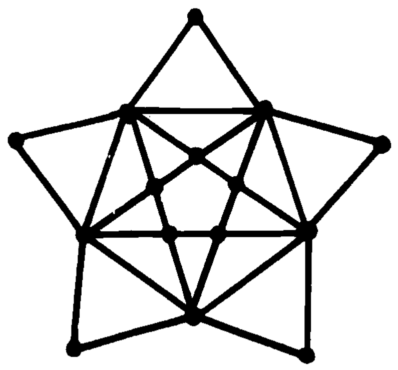


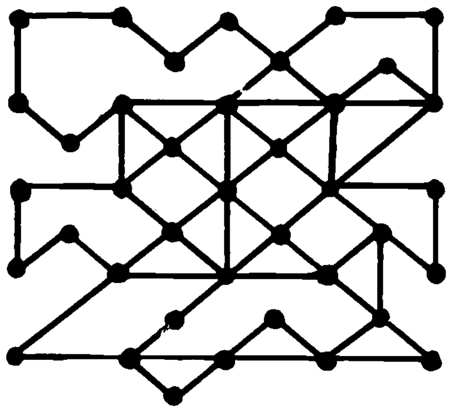


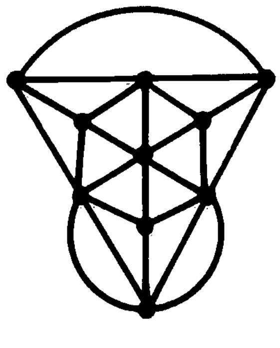


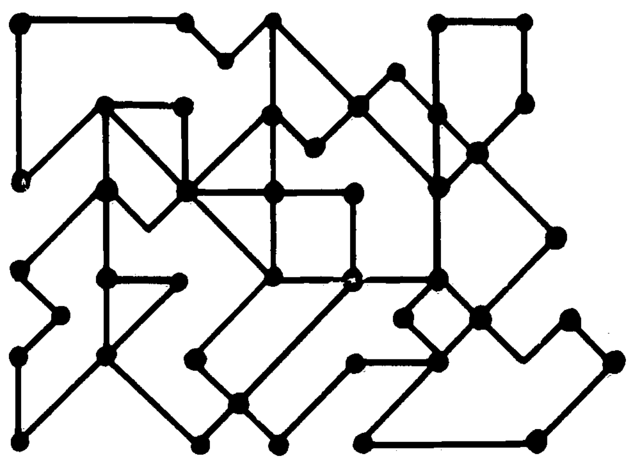












5 Dual Graphs

Geometric puzzles often differ in their apparent objectives as well as in the form in which they are presented. For example, consider Puzzle 2, propounded on page 3 with Figure 4(a) and repeated here with Figure 69. Can you draw a single path crossing each edge of the figure just once without going through a corner?

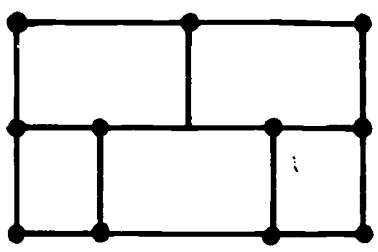


Figure 69

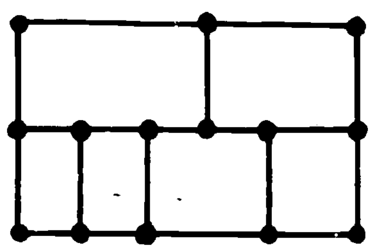


Figure 70

The wording and objective of the puzzle appear somewhat different from our previous considerations with arcs and paths which went through the vertices and traversed the edges of a graph. In this puzzle we are concerned with drawing a curve which "cuts" the edges of a graph. Let us examine this idea of a curve "cutting" the edges of a graph.

A face of a polygonal graph is a distinct region bounded by vertices and edges. If a point lies interior to a face and a second point lies exterior to the face, then it seems "obvious" that any curve connecting the two points must cross an edge or vertex bounding the face. This seemingly obvious (though difficult to prove) assertion is known as the Jordan curve theorem.

For convenience let us restrict our attention to planar and polygonal graphs. Recall that a graph is planar if it is isomorphic to a graph whose edges do not cross or have common points other than at the vertices. A polygonal graph is a connected planar graph such that no single edge completely surrounds a region.

The Jordan Curve Theorem: A continuous nonintersecting closed curve in the plane divides the plane into two regions, an outer and an inner part, such that whenever a point P in the inner part is connected to a point Q in the outer part by a continuous curve, then the two curves must intersect (have a common point). Figure 71 illustrates the theorem.

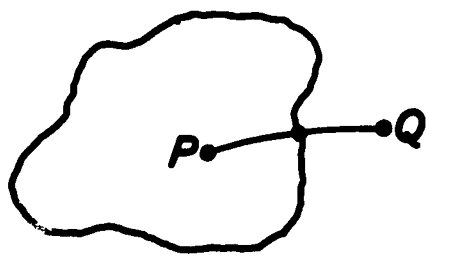


Figure 71

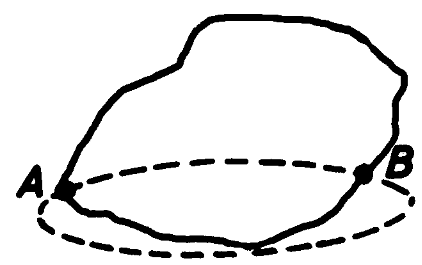


Figure 72

The Jordan curve theorem is useful in studying planar and polygonal graphs with curves passing through them. For example, if A and B are on a continuous closed curve, then any continuous curve connecting A and B which has no other points in common with the closed curve must lie entirely inside or entirely outside of the closed curve, as shown in Figure 72. Puzzle 3 concerning the three tenant farmers and their wells illustrates an application of the Jordan curve theorem.

Recalling how we began with puzzles, consider the problem of drawing a single path crossing each edge of the graph in Figure 69 just once without passing through any vertices. Suppose we start a curving path through the edges of the graph from the exterior as shown in Figure 73. Crossing successive edges just once, we might draw the path P to Q.

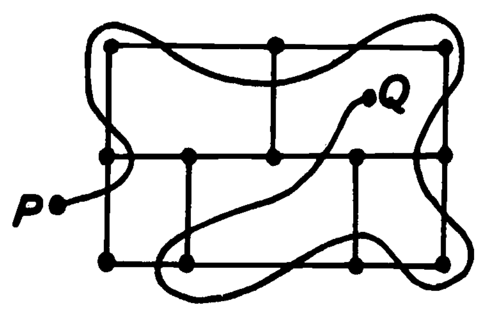
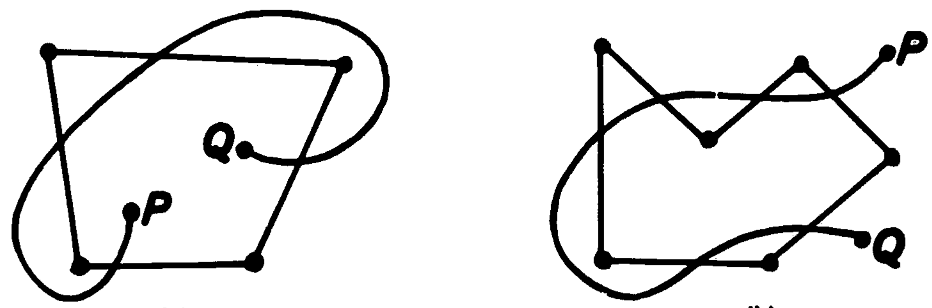


Figure 73

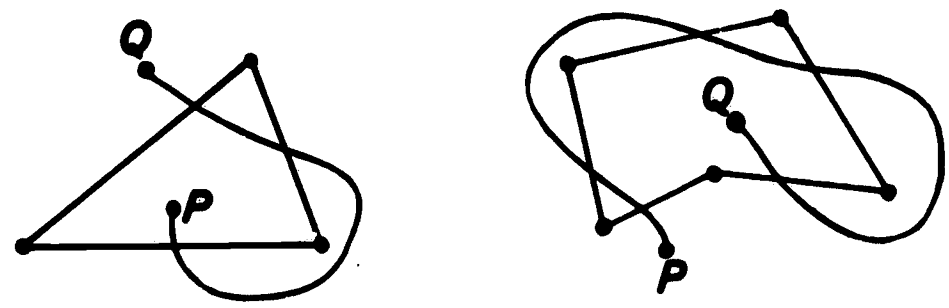
Notice that we have failed to cross one of the edges in the graph. Can we draw another path which will cross each edge? Consider the simplest polygonal graphs consisting of a single "regular" face and the

exterior infinite face. There are two types of faces: faces bounded by an *even* number of edges and faces bounded by an *odd* number of edges. If a face is bounded by an even number of edges, we call it an **even face.** If a face is bounded by an odd number of edges, we call it an **odd face.** Figures 74(a) and 74(b) illustrate two even faces. Figures 75(a) and 75(b) illustrate two odd faces. With these simple graphs we can draw paths and observe the following:



(a) (b)

Figure 74



(a) (b)

Figure 75

A path through an even face must have its initial and terminal points on the same side of the face: that is, both ends inside or both ends outside of the face.

A path through an odd face must have its initial and terminal points on opposite sides of the face: that is, one end must lie on the outside and the other on the inside of the face.

When we construct a path in a polygonal graph, as we cross an edge, we change the number of edges remaining to be crossed from even to odd, or from odd to even, with respect to the path. If the initial point of a path is on the inside of an even face, its terminal point must also be on the inside of the even face. If the initial point of a path is on the outside of an even face, its terminal point must also be on the outside of the even face. The initial and terminal points of a path must be on opposite sides of an odd face.

Now notice that the graph in Figure 69 has three odd faces, so that a path would have to have three ends to cross all edges just once. That is, there is no single path crossing each edge of Figure 69 just once without going through a vertex. The graph in Figure 70, however, has just two odd faces so that a single path with an end interior to each of the odd faces can be drawn. A path for the graph is shown in Figure 76.

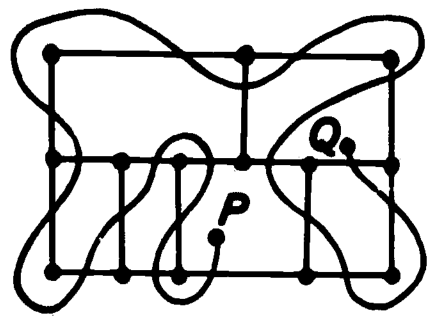


Figure 76

We can summarize four situations as follows:

If a polygonal graph has three or more odd faces, then no complete path crossing each edge just once can be constructed for the graph. See Figures 73 and 77.

If a polygonal graph consists of all even faces, then a complete path crossing each edge just once can be constructed for the graph and, furthermore, this path can be made into a complete cyclic path with a common initial and terminal end. See Figure 78. (The triangular area actually has an even face, since there are four edges, two of them on the same side of the triangle.)

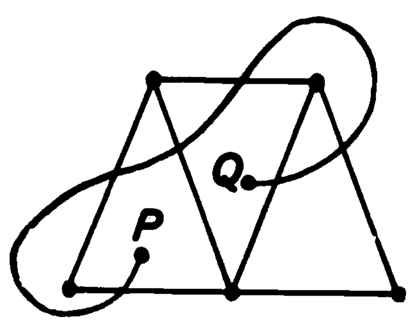


Figure 77

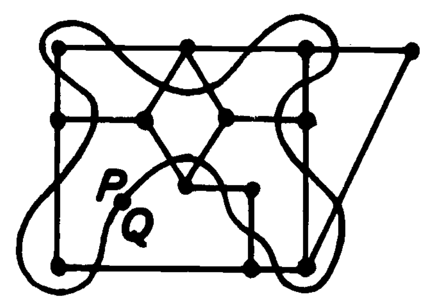


Figure 78

If a polygonal graph has exactly one odd face, then a complete path crossing each edge just once can be constructed with its initial and terminal points on opposite sides of the odd face of the graph. See Figure 79.

If a polygonal graph has exactly two odd faces, then a complete path crossing each edge just once can be constructed with its initial point inside one of the odd faces and its terminal point inside of the other odd face. See Figure 80.

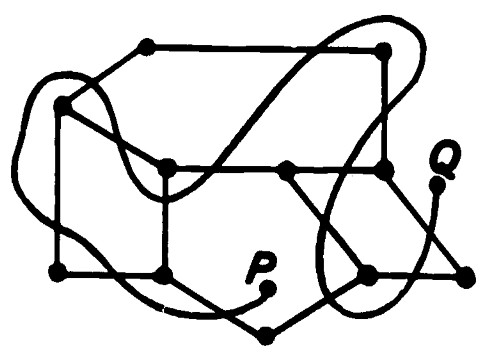


Figure 79

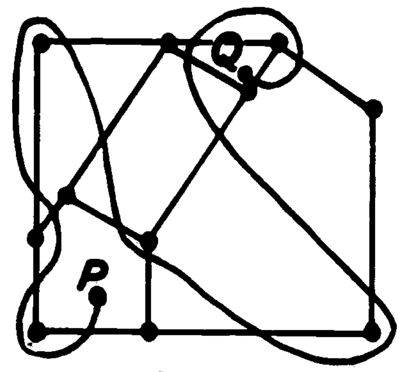


Figure 80

In our discussion of complete paths through polygonal graphs, the similarity to Euler lines and complete paths as discussed in the last section should be apparent. As a matter of fact, an important and close relationship does exist.

For any polygonal graph we can construct a new graph, called its **dual graph**, as follows:

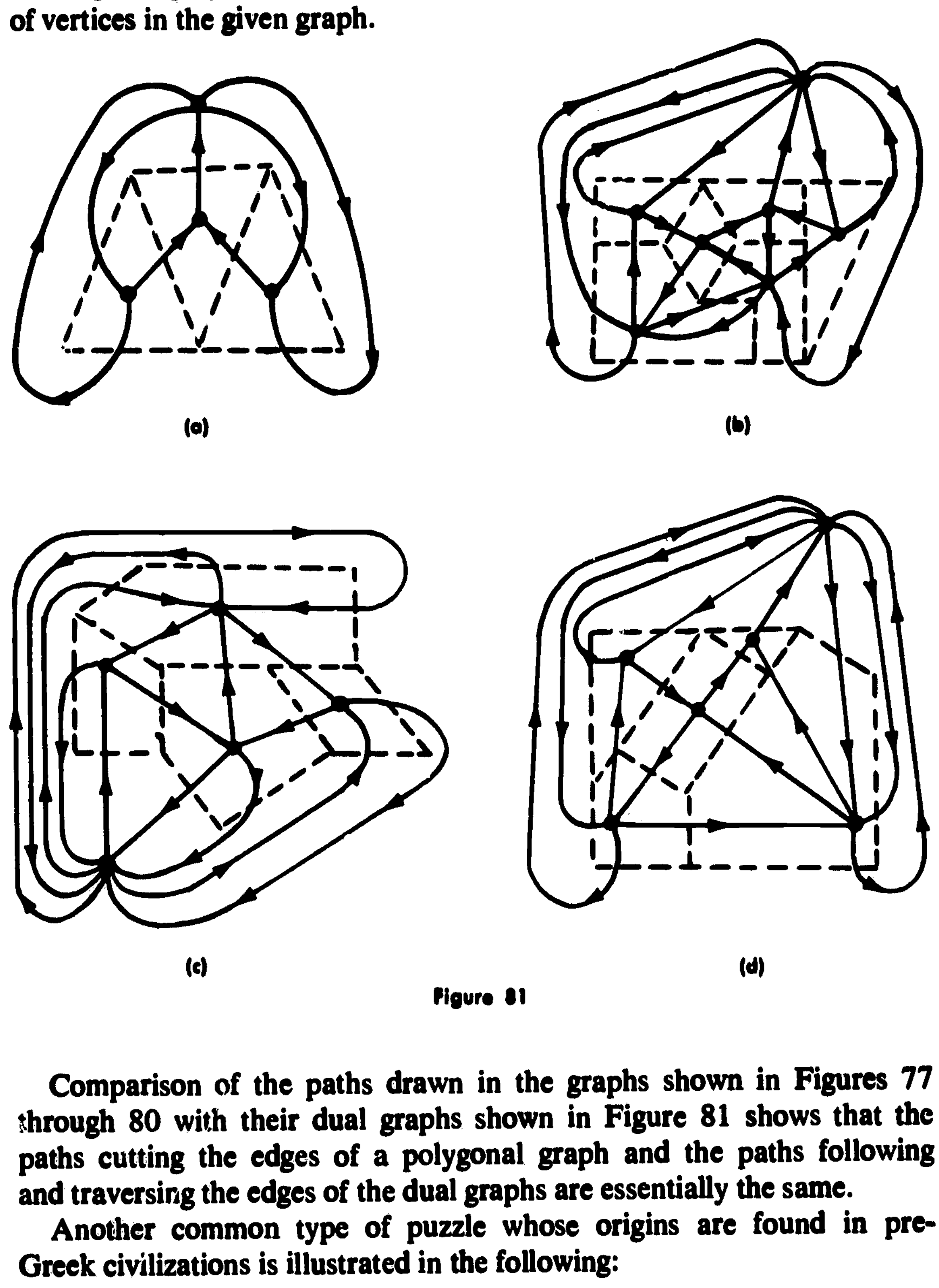
Within each face of the given graph, including the infinite face, we select a single point. If two points are separated by a single edge of the given graph, they are connected by a segment crossing only the one edge of the graph. If there are two or more common edges between faces of the given graph, the points are connected with a segment for each of the common edges. When all of the points have been connected by segments as required, the "new" graph whose vertices are the points introduced in the faces of the given graph and whose edges are the segments drawn crossing the edges of the given graph form the **dual graph** of the given polygonal graph.

Figures 81(a) through 81(d) show the dual graphs of the Figures 77 through 80 respectively. The given graphs are shown in dashed lines.

The dual graph of a polygonal graph is itself a polygonal graph. The dual graph has one vertex for each face of the given graph, including the infinite face. The number of edges incident to a vertex in the dual graph corresponds to the number of boundary edges in the face of the given graph. The local degree of each vertex in the dual graph thus corresponds to the number of edges bounding the face of the given graph. For each edge in the given graph there is a corresponding crossing edge in the dual graph. For each vertex in the given graph there is a

corresponding face in the dual graph. The number of bounding edges to a face in the dual graph corresponds to the local degree of the vertex in the given graph.

A polygonal graph and its dual graph thus have the same number of edges; the number of vertices in the dual graph is the number of faces in the given graph; the number of faces in the dual graph is the number of vertices in the given graph.



(a) (b) (c) (d)

Figure 81

Comparison of the paths drawn in the graphs shown in Figures 77 through 80 with their dual graphs shown in Figure 81 shows that the paths cutting the edges of a polygonal graph and the paths following and traversing the edges of the dual graphs are essentially the same.

Another common type of puzzle whose origins are found in pre-Greek civilizations is illustrated in the following:

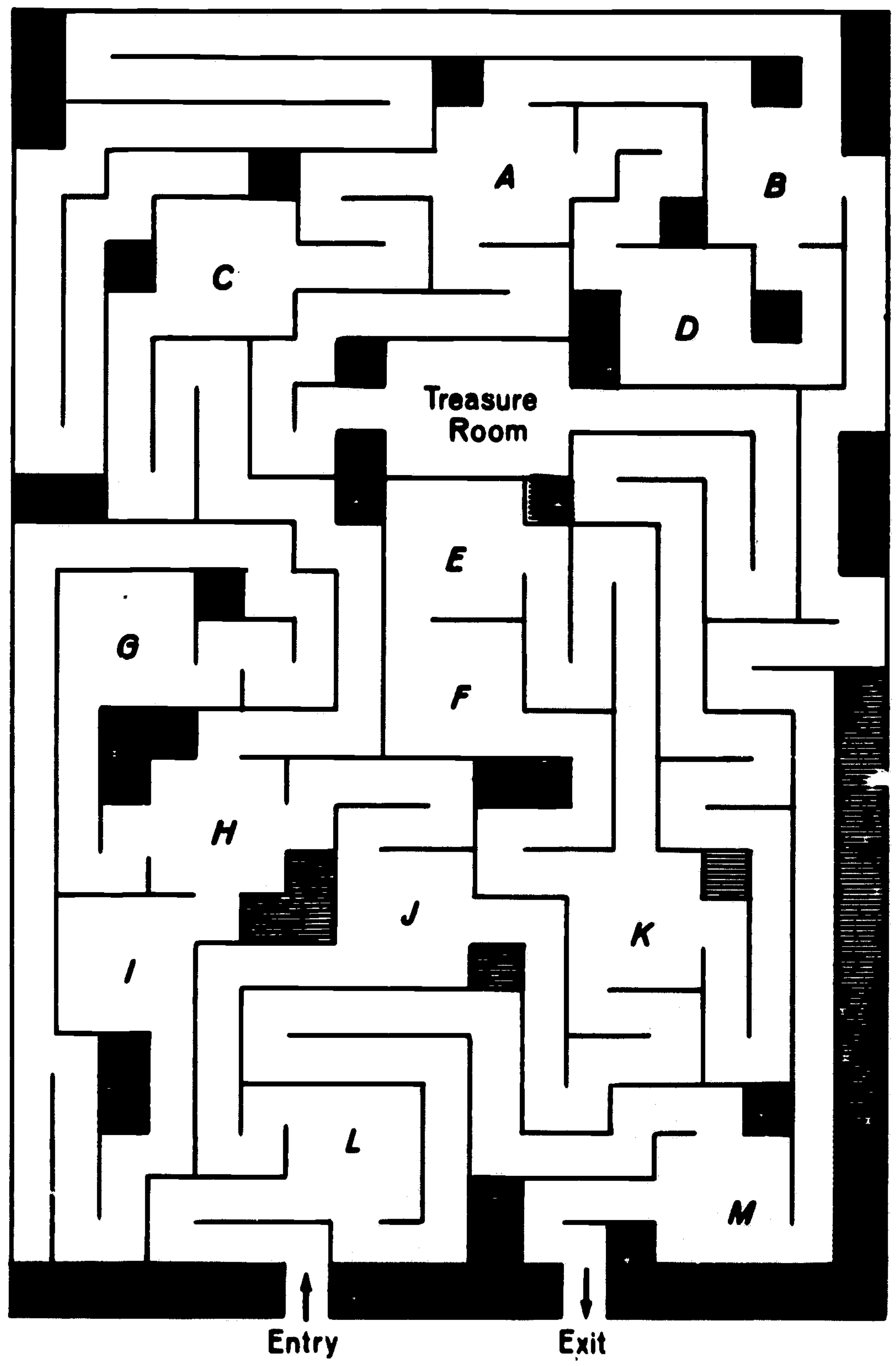


Figure 82

Puzzle 4: A feudal baron kept his treasure hidden in a room of his castle. Not trusting anyone, he had a maze of corridors and doors built so that a corridor could be traveled just once and each room could be entered and exited just once. Given the map shown in Figure 82, can you find a way to the treasure and a way out?

Although such a puzzle may not appear to be related to graphs, notice that we might think of the rooms as vertices and the corridors as edges. The object of the puzzle is then to construct an arc from the entry through the treasure room and thence to the exit. We can represent the puzzle with the graph shown in Figure 83 by selecting points for the rooms and then connecting them with edges which correspond to the corridors.

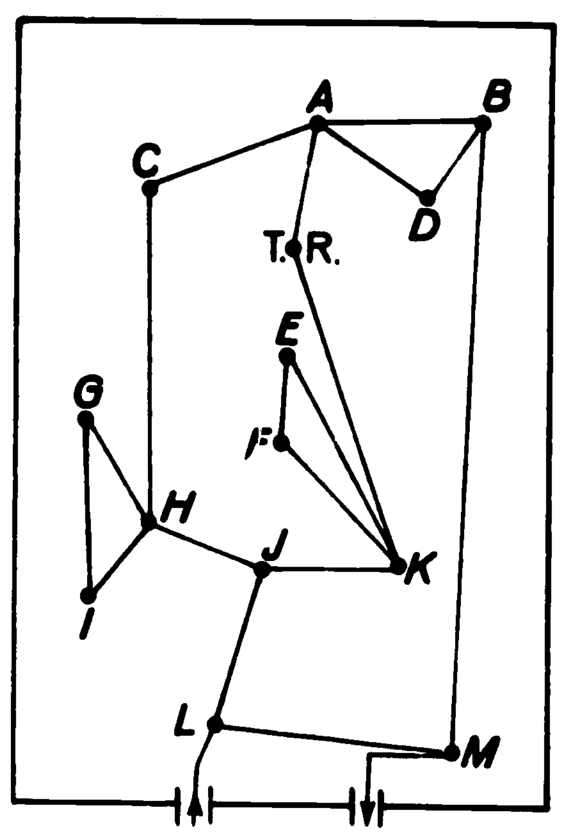


Figure 83

The graphical representation of the puzzle diagram clearly shows the structure of the puzzle. There are two types of "traps" involved in the puzzle: the separating vertices *H* and *K* and the "one-way loops" beginning in the edges *HJ* and *LM*. A solution of the puzzle is the arc *LJK*-T.R. to the treasure room and T.R.-*ABM* to the exit. An alternate exit from the treasure room would be T.R.-*ADBM*.

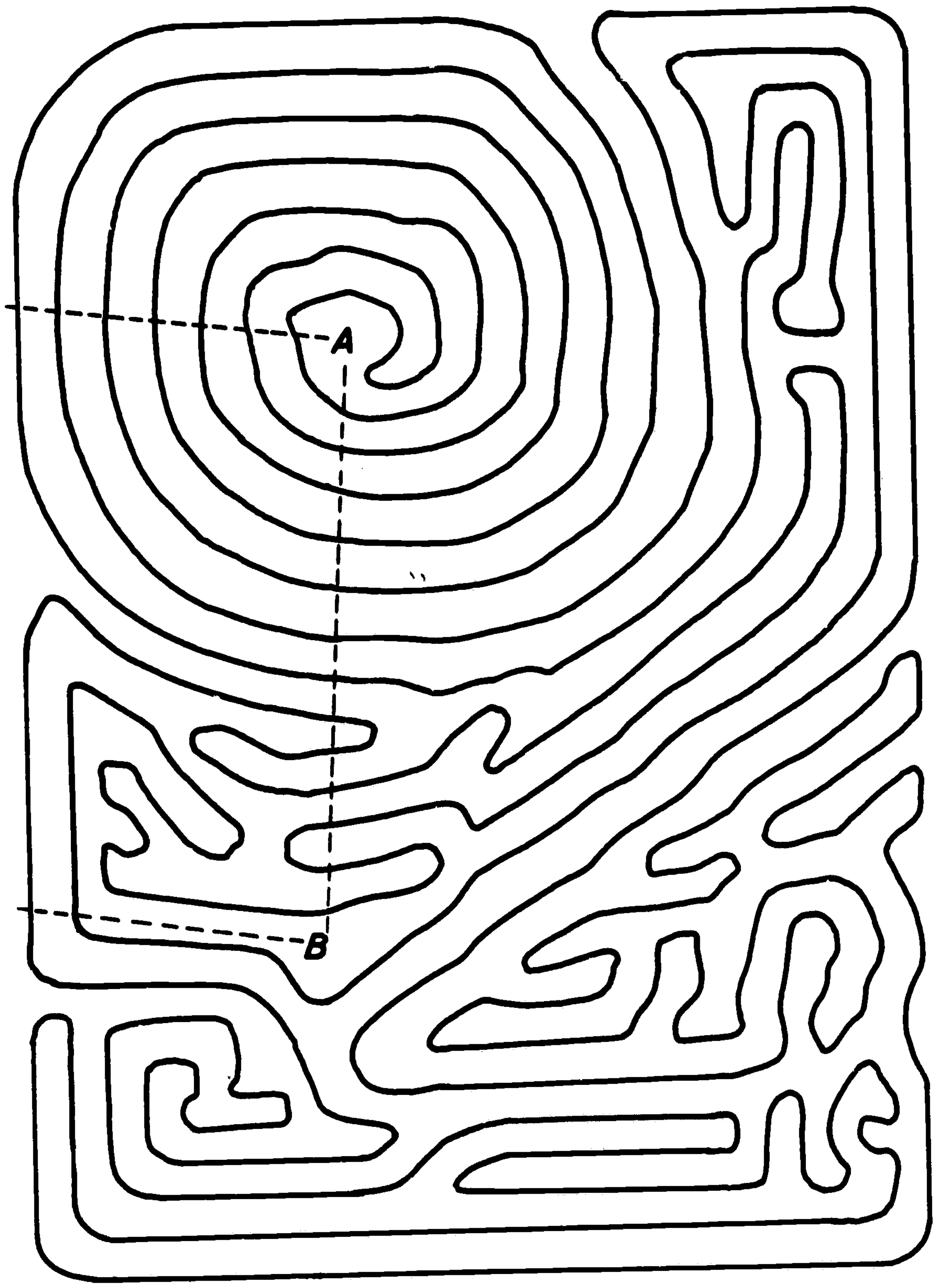


Figure 84

As a final observation, consider the following:

Puzzle 5: A single continuous closed curve winds about as shown in Figure 84. Are the points A and B on the same side of the curve?

This type of maze is quite different from the maze of Puzzle 4. Because a single continuous closed curve is involved, we recall the Jordan curve theorem. The idea is that if we were to "smooth out" the closed curve, and if the points are on the same side of the curve, then we could connect them without intersecting the curve. And if the points are on opposite sides of the curve, there would be a point of intersection with the curve. Now, imagine a point which is clearly on the outside of the closed curve. If we begin with this point and cross a boundary of the curve once, then we must be on the inside of the curve. If we cross the curve twice, we must be on the outside of the curve again. That is, a connection which crosses the curve an odd number of times must have its ends on opposite sides of the curve and a connection which crosses the curve an even number of times must have its ends on the same side of the curve. If we connect the given points A and B in the puzzle and count the number of crossings the connection has with the curve, 13, we can immediately note that the points are on opposite sides of the curve. We might draw a connection from each point directly to the outside of the curve and note that point B is outside the curve while point A is inside the curve.

In this section we have considered two additional ideas with respect to puzzles and graphs: the Jordan curve theorem and the notion of dual graphs. The Jordan curve theorem was useful not only in studying paths cutting closed curves—it also suggested a basic tool for the study of planar graphs. The notion of dual graphs and the concept of isomorphic graphs make available two powerful tools for the interpretation and representation of a variety of situations. To help establish the ideas of this section, you may wish to try the following:

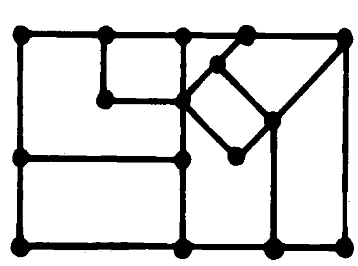


Figure 85

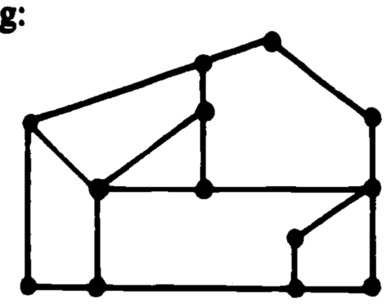


Figure 86

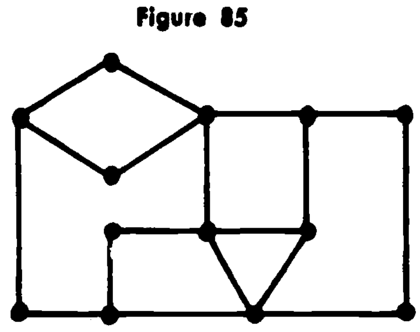


Figure 87

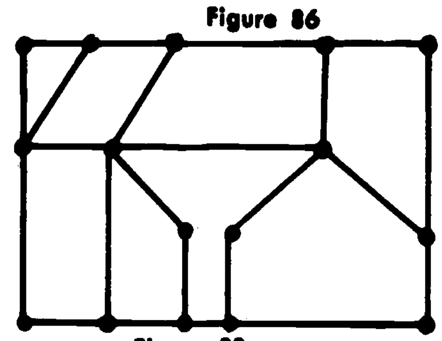


Figure 88

Can you draw a single path crossing each edge of Figure 85 just once without going through a vertex? If so, draw the path. Similarly, consider Figures 86 through 88.

Construct the dual graph for each of the graphs in Figures 85 through 88. If there is an Euler line or complete path for the dual graph, construct it.

Can you find your way through the mazes shown in Figures 89 and 90 without retracing any corridors or entering any room twice?

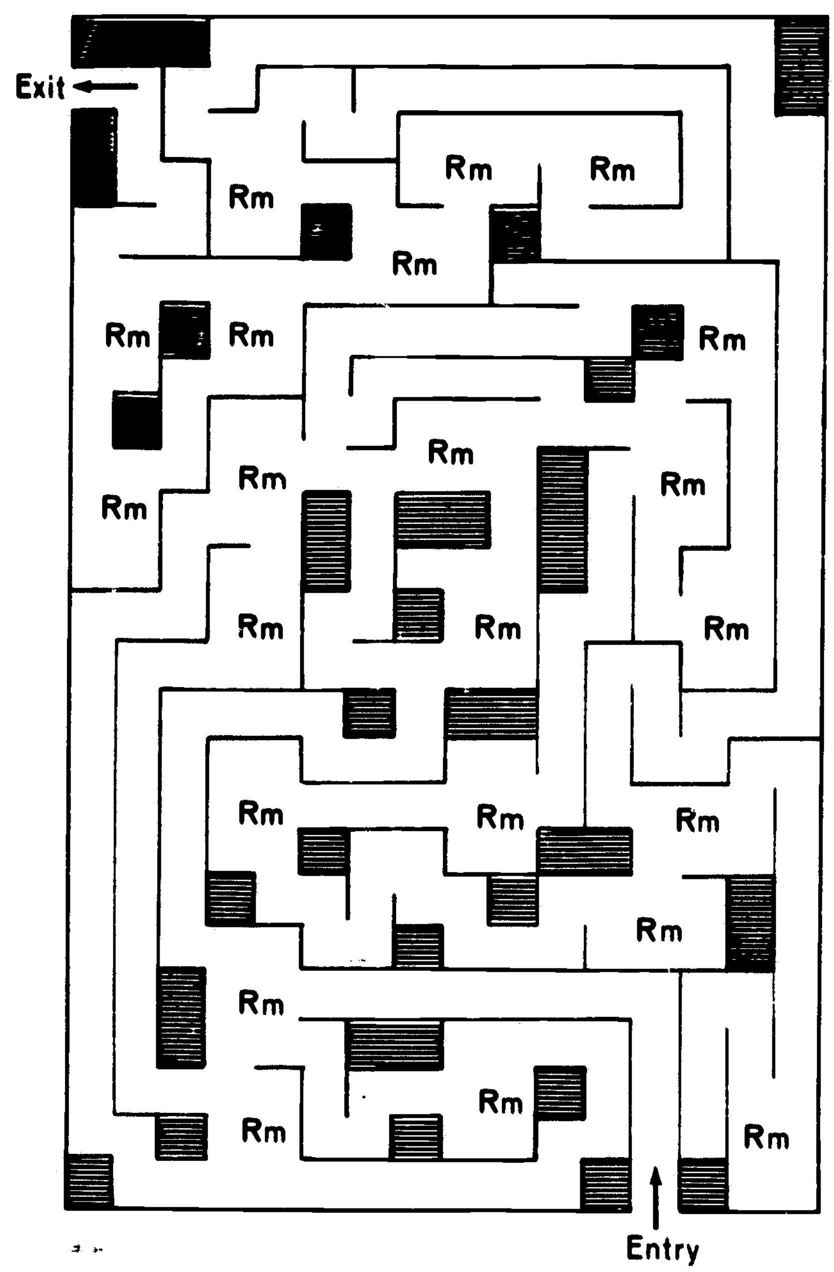


Figure 89

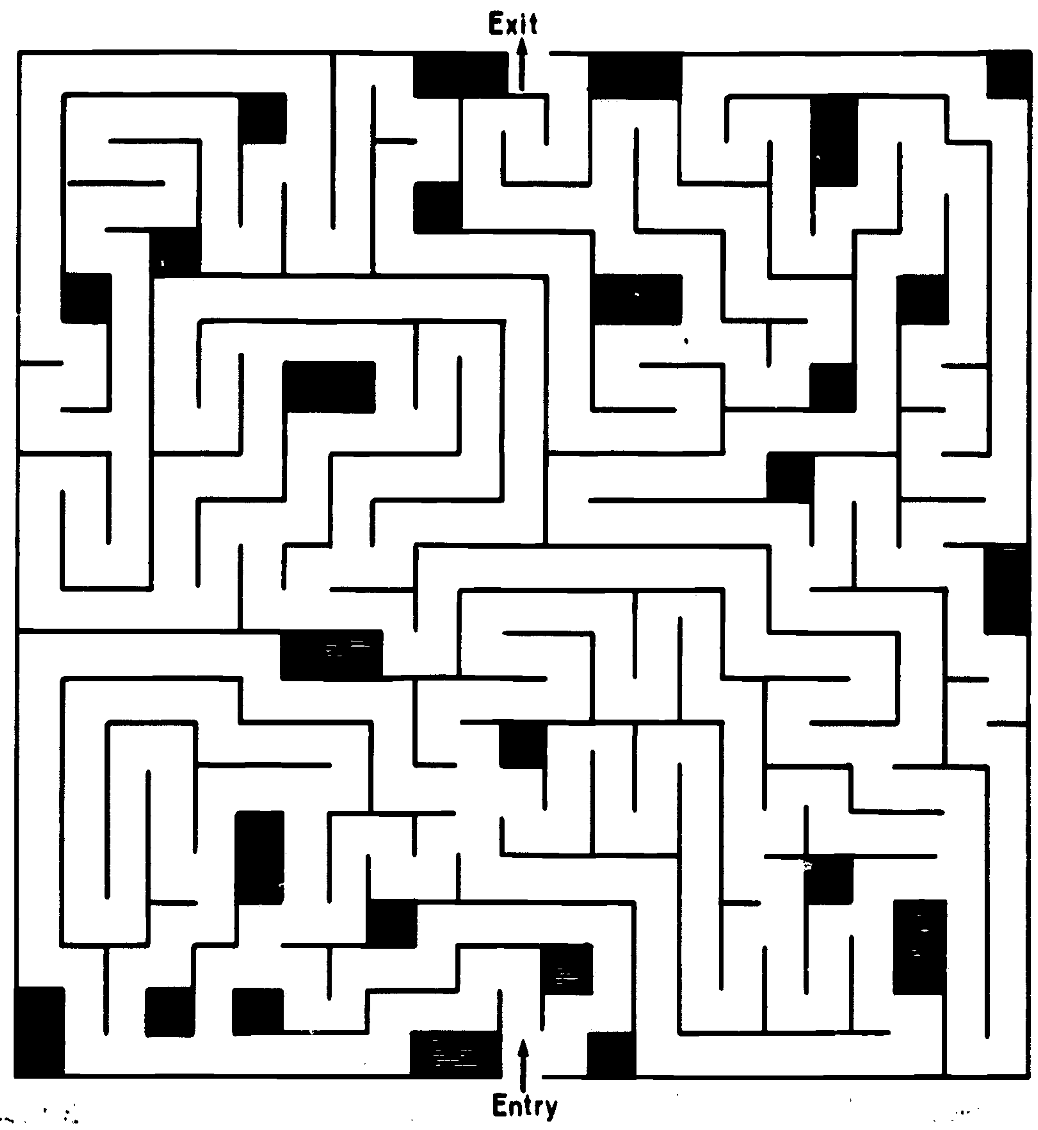


Figure 90

Given the single continuous closed curve shown in Figure 91, at the top of the opposite page, is point *A* interior to the curve? Is point *B* interior to the curve? Are the two points on the same or opposite sides of the curve?

Can you enter and tour the house whose plans are shown in Figure 92 so that you pass through each and every door exactly once before leaving?

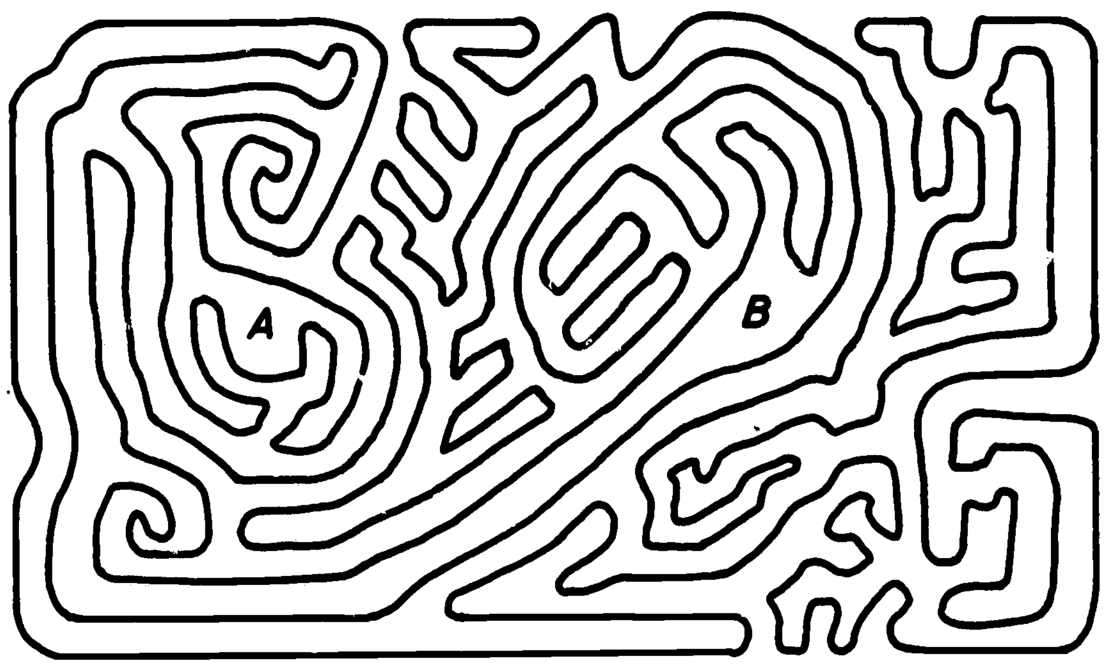


Figure 91

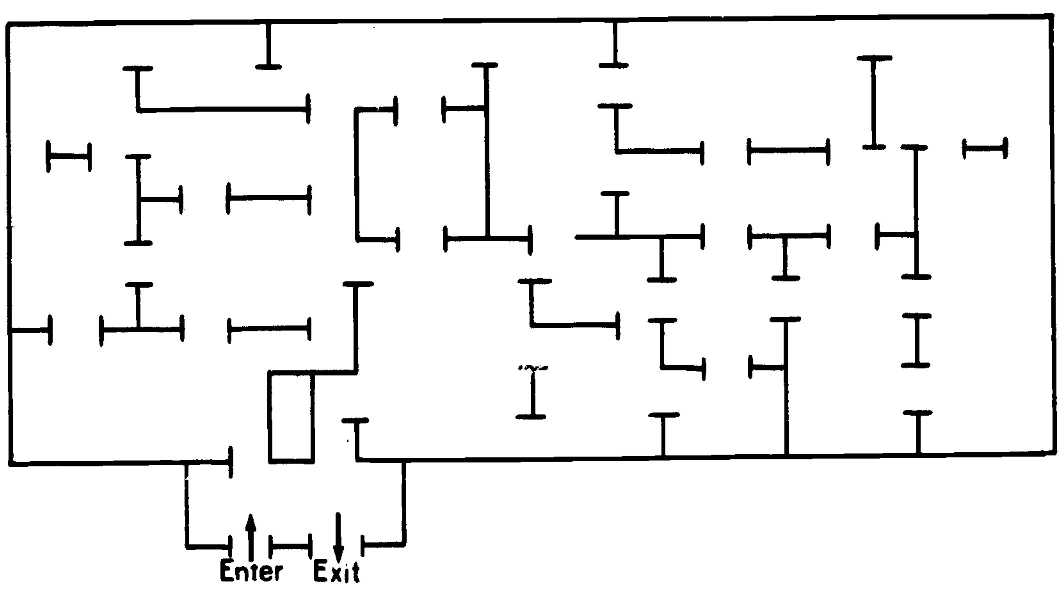


Figure 92

6 Bonus Puzzles

In our brief consideration of puzzles and graphs we have covered only a few of the interpretations and developments possible in this direction. In summary: We have suggested that—

1. A model which carefully describes the characteristics, properties, and relationships assumed to exist in a situation of interest may lead to valuable insights and understanding of the situation.

2. Graphs can be interpreted and used as models in a variety of situations.

3. The ability to construct and recognize objects which have a one-to-one correspondence with a given object is a valuable skill in developing our intuition and in revealing hidden properties of an object.

4. Arcs and paths can be described and their properties examined in terms of the various properties of graphs.

5. The idea of a directed connection is useful in construction as well as interpretations.

6. Euler and Hamilton lines result from definite and distinctive properties of graphs.

7. There are obvious (though difficult to prove) assertions which are fundamental in developing the description of graphs.

8. Given any polygonal graph, we can construct a dual graph which broadens the useful interpretations of graphs.

We began Section 1 with two classic puzzles and it seems fitting to close with a few more. Puzzles, like many problems, are usually not couched in direct easy-to-translate terms. The major hurdle in a puzzle may be to determine the elements and relationships given. Once these, and the objective of the puzzle, have been determined, we are ready to focus on the problem. Most of the following puzzles might actually occur as "real" problems. Some may require ideas not mentioned in this article. However, a bit of ingenuity and determination should lead to adequate solutions of all of them.